

4.1

$$1) |\bar{z}| = |a - bi| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$$

L'exercice 4.4 1) a montré que $z\bar{z} = a^2 + b^2$, d'où suit immédiatement que $\sqrt{z\bar{z}} = \sqrt{a^2 + b^2} = |z|$.

$$2) (a) \text{ Si } z = 0, \text{ alors } |z| = \sqrt{0^2 + 0^2} = 0.$$

(b) Soit $z = a + bi$ un nombre complexe tel que $|z| = 0$.

$$0 \leq |a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = |z| = 0 \text{ impose } |a| = 0, \text{ d'où } a = 0.$$

$$\text{De même, } 0 \leq |b| = \sqrt{b^2} \leq \sqrt{a^2 + b^2} = |z| = 0 \text{ donne } b = 0.$$

Par conséquent, $z = 0 + 0i = 0$.

$$3) |\lambda z| = |\lambda(a + bi)| = |\lambda a + \lambda bi| = \sqrt{(\lambda a)^2 + (\lambda b)^2} = \sqrt{\lambda^2 a^2 + \lambda^2 b^2} = \sqrt{\lambda^2(a^2 + b^2)} = |\lambda| \sqrt{a^2 + b^2} = |\lambda| |z|$$

$$4) |z_1 z_2| = \sqrt{z_1 z_2 \bar{z}_1 \bar{z}_2} = \sqrt{z_1 \bar{z}_1 z_2 \bar{z}_2} = \sqrt{z_1 \bar{z}_1} \sqrt{z_2 \bar{z}_2} = |z_1| |z_2|$$

Autre preuve :

$$\begin{aligned} |z_1 z_2| &= |(a_1 + b_1 i)(a_2 + b_2 i)| = |(a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i| = \\ &= \sqrt{(a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + a_2 b_1)^2} = \\ &= \sqrt{a_1^2 a_2^2 - 2a_1 a_2 b_1 b_2 + b_1^2 b_2^2 + a_1^2 b_2^2 + 2a_1 a_2 b_1 b_2 + a_2^2 b_1^2} = \\ &= \sqrt{a_1^2 a_2^2 + a_1^2 b_2^2 + b_1^2 a_2^2 + b_1^2 b_2^2} = \sqrt{a_1^2(a_2^2 + b_2^2) + a_2^2(a_1^2 + b_1^2)} = \\ &= \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} = \sqrt{a_1^2 + b_1^2} \sqrt{a_2^2 + b_2^2} = |z_1| |z_2| \end{aligned}$$

$$5) 1 = |1| = \left| z \cdot \frac{1}{z} \right| = |z| \left| \frac{1}{z} \right| \text{ donne (en divisant par } |z|) \frac{1}{|z|} = \left| \frac{1}{z} \right|$$

Autre preuve :

$$\left| \frac{1}{z} \right| = \sqrt{\frac{1}{z} \overline{\left(\frac{1}{z} \right)}} = \sqrt{\frac{1}{z} \cdot \frac{1}{\bar{z}}} = \sqrt{\frac{1}{z\bar{z}}} = \frac{\sqrt{1}}{\sqrt{z\bar{z}}} = \frac{1}{|z|}$$

Autre preuve :

$$\begin{aligned} \left| \frac{1}{z} \right| &= \left| \frac{1}{a + bi} \right| = \left| \frac{1(a - bi)}{(a + bi)(a - bi)} \right| = \left| \frac{a - bi}{a^2 + b^2} \right| = \left| \frac{1}{a^2 + b^2} (a - bi) \right| = \\ &= \left| \frac{1}{a^2 + b^2} \right| |a - bi| = \frac{1}{a^2 + b^2} \sqrt{a^2 + (-b)^2} = \frac{\sqrt{a^2 + b^2}}{a^2 + b^2} = \frac{1}{\sqrt{a^2 + b^2}} = \frac{1}{|z|} \end{aligned}$$

$$6) \left| \frac{z_1}{z_2} \right| = \left| z_1 \cdot \frac{1}{z_2} \right| = |z_1| \left| \frac{1}{z_2} \right| = |z_1| \frac{1}{|z_2|} = \frac{|z_1|}{|z_2|}$$