2.6 1) $\alpha_1\,\boldsymbol{v_1}+\alpha_2\,\boldsymbol{v_2}+\alpha_3\,\boldsymbol{v_3}=\mathbf{0} \iff \alpha_1\begin{pmatrix}2\\2\\1\end{pmatrix}+\alpha_2\begin{pmatrix}3\\1\\2\end{pmatrix}+\alpha_3\begin{pmatrix}1\\-5\\2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\left(\begin{array}{ccc|c}2&3&1&0\\2&1&-5&0\\1&2&2&0\end{array}\right)\overset{L_1\leftrightarrow L_3}{\Longrightarrow}\left(\begin{array}{ccc|c}1&2&2&0\\2&1&-5&0\\2&3&1&0\end{array}\right)\overset{\substack{L_2\to L_2-2\,L_1\\L_3\to L_3-2\,L_1}}{\Longrightarrow}$$

$$\left(\begin{array}{ccc|c}1&2&2&0\\0&-3&-9&0\\0&-1&-3&0\end{array}\right)\overset{L_2\leftrightarrow -L_3}{\Longrightarrow}\left(\begin{array}{ccc|c}1&2&2&0\\0&1&3&0\\0&-3&-9&0\end{array}\right)\overset{L_3\to L_3+3\,L_2}{\Longrightarrow}$$

$$\left(\begin{array}{ccc|c}1&2&2&0\\0&1&3&0\\0&0&0&0\end{array}\right)\overset{L_1\to L_1-2\,L_2}{\Longrightarrow}\left(\begin{array}{ccc|c}1&0&-4&0\\0&1&3&0\\0&0&0&0\end{array}\right)$$

$$\begin{cases}\alpha_1 \quad = \quad 4\,\alpha\\ \quad \alpha_2 \quad = -3\,\alpha \quad \text{où } \alpha\in\mathbb{R}\\ \qquad \alpha_3 = \; = \alpha\end{cases}$$

Il y a une infinité de solutions : la famille $\boldsymbol{v_1},\boldsymbol{v_2},\boldsymbol{v_3}$ est liée.

Si l'on choisit $\alpha=1$, alors $\alpha_1=4$, $\alpha_2=-3$ et $\alpha_3=1$:

$$4\,\boldsymbol{v_1}-3\,\boldsymbol{v_2}+\boldsymbol{v_3}=\mathbf{0}$$

2) $\alpha_1\,\boldsymbol{v_1}+\alpha_2\,\boldsymbol{v_2}=\mathbf{0} \iff \alpha_1\begin{pmatrix}2\\-1\\3\end{pmatrix}+\alpha_2\begin{pmatrix}-1\\2\\3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\left(\begin{array}{cc|c}2&-1&0\\-1&2&0\\3&3&0\end{array}\right)\overset{L_1\leftrightarrow -L_2}{\Longrightarrow}\left(\begin{array}{cc|c}1&-2&0\\2&-1&0\\3&3&0\end{array}\right)\overset{\substack{L_2\to L_2-2\,L_1\\L_3\to L_3-3\,L_1}}{\Longrightarrow}\left(\begin{array}{cc|c}1&-2&0\\0&3&0\\0&9&0\end{array}\right)$$

$$\overset{\substack{L_2\to 1/3\,L_2\\L_3\to 1/9\,L_3}}{\Longrightarrow}\left(\begin{array}{cc|c}1&-2&0\\0&1&0\\0&1&0\end{array}\right)\overset{\substack{L_1\to L_1+2\,L_2\\L_3\to L_3-L_2}}{\Longrightarrow}\left(\begin{array}{cc|c}1&0&0\\0&1&0\\0&0&0\end{array}\right)\Longrightarrow\begin{cases}\alpha_1 \quad =0\\ \quad \alpha_2=0\end{cases}$$

Il y a une solution unique : la famille $\boldsymbol{v_1},\boldsymbol{v_2}$ est libre.

3) $\alpha_1\,\boldsymbol{v_1}+\alpha_2\,\boldsymbol{v_2}+\alpha_3\,\boldsymbol{v_3}+\alpha_4\,\boldsymbol{v_4}=\mathbf{0} \iff$
$\alpha_1\begin{pmatrix}-2\\3\\7\end{pmatrix}+\alpha_2\begin{pmatrix}4\\-1\\5\end{pmatrix}+\alpha_3\begin{pmatrix}3\\1\\3\end{pmatrix}+\alpha_4\begin{pmatrix}5\\0\\2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\left(\begin{array}{cccc|c}-2&4&3&5&0\\3&-1&1&0&0\\7&5&3&2&0\end{array}\right)\overset{\substack{L_2\to 2\,L_2+3\,L_1\\L_3\to 2\,L_3+7\,L_1}}{\Longrightarrow}\left(\begin{array}{cccc|c}-2&4&3&5&0\\0&10&11&15&0\\0&38&27&39&0\end{array}\right)\overset{L_3\to 5\,L_3-19\,L_2}{\Longrightarrow}$$

$$\left(\begin{array}{cccc|c}-2&4&3&5&0\\0&10&11&15&0\\0&0&-74&-90&0\end{array}\right)\overset{L_3\to -1/2\,L_3}{\Longrightarrow}\left(\begin{array}{cccc|c}-2&4&3&5&0\\0&10&11&15&0\\0&0&37&45&0\end{array}\right)\overset{\substack{L_1\to 37\,L_1-3\,L_3\\L_2\to 37\,L_2-11\,L_3}}{\Longrightarrow}$$

$$\left(\begin{array}{cccc|c} -74 & 148 & 0 & 50 & 0 \\ 0 & 370 & 0 & 60 & 0 \\ 0 & 0 & 37 & 45 & 0 \end{array}\right) \underset{\substack{L_1 \to -1/2\, L_1 \\ L_2 \to 1/10\, L_2}}{\Longrightarrow} \left(\begin{array}{cccc|c} 37 & -74 & 0 & -25 & 0 \\ 0 & 37 & 0 & 6 & 0 \\ 0 & 0 & 37 & 45 & 0 \end{array}\right) \overset{L_1 \to L_1 + 2\, L_2}{\Longrightarrow}$$

$$\left(\begin{array}{cccc|c} 37 & 0 & 0 & -13 & 0 \\ 0 & 37 & 0 & 6 & 0 \\ 0 & 0 & 37 & 45 & 0 \end{array}\right) \underset{\substack{L_1 \to 1/37\, L_1 \\ L_2 \to 1/37\, L_2 \\ L_3 \to 1/37\, L_3}}{\Longrightarrow} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{13}{37} & 0 \\ 0 & 1 & 0 & \frac{6}{37} & 0 \\ 0 & 0 & 1 & \frac{45}{37} & 0 \end{array}\right)$$

$$\begin{cases} \alpha_1 = 13\,\alpha \\ \alpha_2 = -6\,\alpha \\ \alpha_3 = -45\,\alpha \\ \alpha_4 = 37\,\alpha \end{cases} \text{où } \alpha \in \mathbb{R}$$

Il y a une infinité de solutions : la famille $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}$ est liée.

Si l'on choisit $\alpha = 1$, alors $\alpha_1 = 13$, $\alpha_2 = -6$, $\alpha_3 = -45$ et $\alpha_4 = 37$:

$$13\,\boldsymbol{v_1} - 6\,\boldsymbol{v_2} - 45\,\boldsymbol{v_3} + 37\,\boldsymbol{v_4} = \mathbf{0}$$