**2.7** 1) $\alpha_1\,\boldsymbol{v_1}+\alpha_2\,\boldsymbol{v_2}+\alpha_3\,\boldsymbol{v_3}=\boldsymbol{0} \iff \alpha_1\begin{pmatrix}-1\\1\\2\\1\end{pmatrix}+\alpha_2\begin{pmatrix}3\\2\\2\\4\end{pmatrix}+\alpha_3\begin{pmatrix}2\\3\\1\\-1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

$$\left(\begin{array}{ccc|c}-1&3&2&0\\1&2&3&0\\2&2&1&0\\1&4&-1&0\end{array}\right)\overset{\substack{L_2\to L_2+L_1\\L_3\to L_3+2\,L_1\\L_4\to L_4+L_1}}{\Longrightarrow}\left(\begin{array}{ccc|c}-1&3&2&0\\0&5&5&0\\0&8&5&0\\0&7&1&0\end{array}\right)\overset{L_2\to 1/5\,L_2}{\Longrightarrow}$$

$$\left(\begin{array}{ccc|c}-1&3&2&0\\0&1&1&0\\0&8&5&0\\0&7&1&0\end{array}\right)\overset{\substack{L_3\to L_3-8\,L_2\\L_4\to L_4-7\,L_2}}{\Longrightarrow}\left(\begin{array}{ccc|c}-1&3&2&0\\0&1&1&0\\0&0&-3&0\\0&0&-6&0\end{array}\right)\overset{\substack{L_3\to -1/3\,L_3\\L_4\to -1/6\,L_4}}{\Longrightarrow}$$

$$\left(\begin{array}{ccc|c}-1&3&2&0\\0&1&1&0\\0&0&1&0\\0&0&1&0\end{array}\right)\overset{\substack{L_1\to L_1-2\,L_3\\L_2\to L_2-L_3\\L_4\to L_4-L_3}}{\Longrightarrow}\left(\begin{array}{ccc|c}-1&3&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right)\overset{L_1\to -L_1+3\,L_2}{\Longrightarrow}$$

$$\left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right)\Longrightarrow\begin{cases}\alpha_1 & =0\\ \quad\alpha_2 & =0\\ \qquad\alpha_3 & =0\end{cases}$$

Il y a une solution unique : la famille $\boldsymbol{v_1},\boldsymbol{v_2},\boldsymbol{v_3}$ est libre.

2) $\alpha_1\,\boldsymbol{v_1}+\alpha_2\,\boldsymbol{v_2}+\alpha_3\,\boldsymbol{v_3}+\alpha_4\,\boldsymbol{v_4}=\boldsymbol{0} \iff$
$\alpha_1\begin{pmatrix}0\\0\\0\\1\end{pmatrix}+\alpha_2\begin{pmatrix}0\\0\\2\\1\end{pmatrix}+\alpha_3\begin{pmatrix}0\\3\\2\\1\end{pmatrix}+\alpha_4\begin{pmatrix}4\\3\\2\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

$$\left(\begin{array}{cccc|c}0&0&0&4&0\\0&0&3&3&0\\0&2&2&2&0\\1&1&1&1&0\end{array}\right)\overset{\substack{L_1\to 1/4\,L_1\\L_2\to 1/3\,L_2\\L_3\to 1/2\,L_3}}{\Longrightarrow}\left(\begin{array}{cccc|c}0&0&0&1&0\\0&0&1&1&0\\0&1&1&1&0\\1&1&1&1&0\end{array}\right)\overset{\substack{L_1\leftrightarrow L_4\\L_2\leftrightarrow L_3}}{\Longrightarrow}$$

$$\left(\begin{array}{cccc|c}1&1&1&1&0\\0&1&1&1&0\\0&0&1&1&0\\0&0&0&1&0\end{array}\right)\overset{\substack{L_1\to L_1-L_4\\L_2\to L_2-L_4\\L_3\to L_3-L_4}}{\Longrightarrow}\left(\begin{array}{cccc|c}1&1&1&0&0\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&0\end{array}\right)\overset{\substack{L_1\to L_1-L_3\\L_2\to L_2-L_3}}{\Longrightarrow}$$

$$\left(\begin{array}{cccc|c}1&1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{array}\right)\overset{L_1\to L_1-L_2}{\Longrightarrow}\left(\begin{array}{cccc|c}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{array}\right)\Longrightarrow\begin{cases}\alpha_1 & =0\\ \quad\alpha_2 & =0\\ \qquad\alpha_3 & =0\\ \qquad\quad\alpha_4 & =0\end{cases}$$

Il y a une solution unique : la famille $\boldsymbol{v_1},\boldsymbol{v_2},\boldsymbol{v_3},\boldsymbol{v_4}$ est libre.

3) $\alpha_1\,\boldsymbol{v_1} + \alpha_2\,\boldsymbol{v_2} + \alpha_3\,\boldsymbol{v_3} + \alpha_4\,\boldsymbol{v_4} = \mathbf{0} \quad \Longleftrightarrow$

$$\alpha_1 \begin{pmatrix} 3 \\ -1 \\ 1 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 3 \\ 1 \\ -1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix} + \alpha_4 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 3 & -1 & 1 & -1 & 0 \\ -1 & 3 & 1 & -1 & 0 \\ 1 & 1 & 3 & 1 & 0 \\ -1 & -1 & 1 & 3 & 0 \end{array}\right) \overset{\mathrm{L}_1 \leftrightarrow \mathrm{L}_3}{\Longrightarrow} \left(\begin{array}{cccc|c} 1 & 1 & 3 & 1 & 0 \\ -1 & 3 & 1 & -1 & 0 \\ 3 & -1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 3 & 0 \end{array}\right) \overset{\substack{\mathrm{L}_2 \to \mathrm{L}_2 + \mathrm{L}_1 \\ \mathrm{L}_3 \to \mathrm{L}_3 - 3\,\mathrm{L}_1 \\ \mathrm{L}_4 \to \mathrm{L}_4 + \mathrm{L}_1}}{\Longrightarrow}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 3 & 1 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & -4 & -8 & -4 & 0 \\ 0 & 0 & 4 & 4 & 0 \end{array}\right) \overset{\substack{\mathrm{L}_2 \to 1/4\,\mathrm{L}_2 \\ \mathrm{L}_3 \to -1/4\,\mathrm{L}_3 \\ \mathrm{L}_4 \to 1/4\,\mathrm{L}_4}}{\Longrightarrow} \left(\begin{array}{cccc|c} 1 & 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \overset{\mathrm{L}_3 \to \mathrm{L}_3 - \mathrm{L}_2}{\Longrightarrow}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \overset{\mathrm{L}_4 \to \mathrm{L}_4 - \mathrm{L}_3}{\Longrightarrow} \left(\begin{array}{cccc|c} 1 & 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \overset{\substack{\mathrm{L}_1 \to \mathrm{L}_1 - 3\,\mathrm{L}_3 \\ \mathrm{L}_2 \to \mathrm{L}_2 - \mathrm{L}_3}}{\Longrightarrow}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \overset{\mathrm{L}_1 \to \mathrm{L}_1 - \mathrm{L}_2}{\Longrightarrow} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left\{\begin{array}{ccccrr} \alpha_1 & & & & = & \alpha \\ & \alpha_2 & & & = & \alpha \\ & & \alpha_3 & & = & -\alpha \\ & & & \alpha_4 & = & \alpha \end{array}\right. \text{ où } \alpha \in \mathbb{R}$$

Il y a une infinité de solutions : la famille $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}$ est liée.

En choisissant $\alpha = 1$, on obtient $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = -1$ et $\alpha_4 = 1$ :

$$\boldsymbol{v_1} + \boldsymbol{v_2} - \boldsymbol{v_3} + \boldsymbol{v_4} = \mathbf{0}$$