3.1 1) (a) i. $f(\boldsymbol{v}+\boldsymbol{w}) = f\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = f\left(\begin{pmatrix} v_1+w_1 \\ v_2+w_2 \end{pmatrix}\right)$

$= \begin{pmatrix} (v_1+w_1)+(v_2+w_2) \\ v_1+w_1 \end{pmatrix} = \begin{pmatrix} v_1+w_1+v_2+w_2 \\ v_1+w_1 \end{pmatrix}$

ii. $f(\boldsymbol{v})+f(\boldsymbol{w}) = f\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) + f\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = \begin{pmatrix} v_1+v_2 \\ v_1 \end{pmatrix} + \begin{pmatrix} w_1+w_2 \\ w_1 \end{pmatrix}$

$= \begin{pmatrix} v_1+v_2+w_1+w_2 \\ v_1+w_1 \end{pmatrix}$

Puisque les deux résultats sont égaux, on a que $f(\boldsymbol{v}+\boldsymbol{w}) = f(\boldsymbol{v})+f(\boldsymbol{w})$.

(b) i. $f(\alpha\,\boldsymbol{v}) = f\left(\alpha\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = f\left(\begin{pmatrix} \alpha\,v_1 \\ \alpha\,v_2 \end{pmatrix}\right) = \begin{pmatrix} \alpha\,v_1+\alpha\,v_2 \\ \alpha\,v_1 \end{pmatrix}$

ii. $\alpha\,f(\boldsymbol{v}) = \alpha\,f\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \alpha\begin{pmatrix} v_1+v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} \alpha\,(v_1+v_2) \\ \alpha\,v_1 \end{pmatrix} = \begin{pmatrix} \alpha\,v_1+\alpha\,v_2 \\ \alpha\,v_1 \end{pmatrix}$

Comme les résultats sont identiques, on a que $f(\alpha\,\boldsymbol{v}) = \alpha\,f(\boldsymbol{v})$.

L'application f est bien linéaire.

2) (a) i. $f(\boldsymbol{v}+\boldsymbol{w}) = f\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = f\left(\begin{pmatrix} v_1+w_1 \\ v_2+w_2 \end{pmatrix}\right)$

$= \begin{pmatrix} (v_1+w_1)+1 \\ 2\,(v_2+w_2) \\ (v_1+w_1)+(v_2+w_2) \end{pmatrix} = \begin{pmatrix} v_1+w_1+1 \\ 2\,v_2+2\,w_2 \\ v_1+w_1+v_2+w_2 \end{pmatrix}$

ii. $f(\boldsymbol{v})+f(\boldsymbol{w}) = f\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) + f\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = \begin{pmatrix} v_1+1 \\ 2\,v_2 \\ v_1+v_2 \end{pmatrix} + \begin{pmatrix} w_1+1 \\ 2\,w_2 \\ w_1+w_2 \end{pmatrix}$

$= \begin{pmatrix} v_1+w_1+2 \\ 2\,v_2+2\,w_2 \\ v_1+v_2+w_1+w_2 \end{pmatrix}$

Les premières composantes diffèrent : $f(\boldsymbol{v}+\boldsymbol{w}) \neq f(\boldsymbol{v})+f(\boldsymbol{w})$.

Inutile de tester la seconde propriété : l'application f n'est pas linéaire.

3) (a) i. $f(\boldsymbol{v}+\boldsymbol{w}) = f\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\right) = f\left(\begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ v_3+w_3 \end{pmatrix}\right)$

$= 2\,(v_1+w_1) - 3\,(v_2+w_2) + 4\,(v_3+w_3)$

$= 2\,v_1 + 2\,w_1 - 3\,v_2 - 3\,w_2 + 4\,v_3 + 4\,w_3$

ii. $f(\boldsymbol{v})+f(\boldsymbol{w}) = f\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) + f\left(\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\right)$

$= 2\,v_1 - 3\,v_2 + 4\,v_3 + 2\,w_1 - 3\,w_2 + 4\,w_3$

Puisque les deux résultats sont égaux, on a que $f(\boldsymbol{v}+\boldsymbol{w}) = f(\boldsymbol{v})+f(\boldsymbol{w})$.

(b) i. $f(\alpha\,\boldsymbol{v}) = f\left(\alpha\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) = f\left(\begin{pmatrix} \alpha\,v_1 \\ \alpha\,v_2 \\ \alpha\,v_3 \end{pmatrix}\right) = 2\,\alpha\,v_1 - 3\,\alpha\,v_2 + 4\,\alpha\,v_3$

ii. $\alpha\, f(\boldsymbol{v}) = \alpha\, f\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) = \alpha\,(2\,v_1 - 3\,v_2 + 4\,v_3) = 2\,\alpha\,v_1 - 3\,\alpha\,v_2 + 4\,\alpha\,v_3$

Comme les résultats sont identiques, on a que $f(\alpha\,\boldsymbol{v}) = \alpha\, f(\boldsymbol{v})$.

L'application f est bien linéaire.

4) (a) i. $f(\boldsymbol{v} + \boldsymbol{w}) = f\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = f\left(\begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}\right) = \begin{pmatrix} \sin(v_1 + w_1) \\ v_2 + w_2 \end{pmatrix}$

ii. $f(\boldsymbol{v}) + f(\boldsymbol{w}) = f\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) + f\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = \begin{pmatrix} \sin(v_1) \\ v_2 \end{pmatrix} + \begin{pmatrix} \sin(w_1) \\ w_2 \end{pmatrix}$

$= \begin{pmatrix} \sin(v_1) + \sin(w_1) \\ v_2 + w_2 \end{pmatrix}$

Les premières composantes diffèrent : $f(\boldsymbol{v} + \boldsymbol{w}) \neq f(\boldsymbol{v}) + f(\boldsymbol{w})$.

Inutile de tester la seconde propriété : l'application f n'est pas linéaire.

5) (a) i. $f(\boldsymbol{v} + \boldsymbol{w}) = f\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = f\left(\begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}\right)$

$= \begin{pmatrix} (v_1 + w_1)^2 \\ (v_1 + w_1) + (v_2 + w_2) \end{pmatrix} = \begin{pmatrix} v_1^2 + 2\,v_1\,w_1 + w_1^2 \\ v_1 + w_1 + v_2 + w_2 \end{pmatrix}$

ii. $f(\boldsymbol{v}) + f(\boldsymbol{w}) = f\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) + f\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = \begin{pmatrix} v_1^2 \\ v_1 + v_2 \end{pmatrix} + \begin{pmatrix} w_1^2 \\ w_1 + w_2 \end{pmatrix}$

$= \begin{pmatrix} v_1^2 + w_1^2 \\ v_1 + v_2 + w_1 + w_2 \end{pmatrix}$

Les premières composantes diffèrent : $f(\boldsymbol{v} + \boldsymbol{w}) \neq f(\boldsymbol{v}) + f(\boldsymbol{w})$.

Inutile de tester la seconde propriété : l'application f n'est pas linéaire.

6) (a) i. $f(\boldsymbol{v} + \boldsymbol{w}) = f\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\right) = f\left(\begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix}\right)$

$= \begin{pmatrix} 0 \\ v_1 + w_1 \\ 2\,(v_1 + w_1) \end{pmatrix} = \begin{pmatrix} 0 \\ v_1 + w_1 \\ 2\,v_1 + 2\,w_1 \end{pmatrix}$

ii. $f(\boldsymbol{v}) + f(\boldsymbol{w}) = f\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) + f\left(\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\right)$

$= \begin{pmatrix} 0 \\ v_1 \\ 2\,v_1 \end{pmatrix} + \begin{pmatrix} 0 \\ w_1 \\ 2\,w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ v_1 + w_1 \\ 2\,v_1 + 2\,w_1 \end{pmatrix}$

Puisque les deux résultats sont égaux, on a que $f(\boldsymbol{v}+\boldsymbol{w}) = f(\boldsymbol{v})+f(\boldsymbol{w})$.

(b) i. $f(\alpha\,\boldsymbol{v}) = f\left(\alpha \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) = f\left(\begin{pmatrix} \alpha\,v_1 \\ \alpha\,v_2 \\ \alpha\,v_3 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ \alpha\,v_1 \\ 2\,\alpha\,v_1 \end{pmatrix}$

ii. $\alpha\, f(\boldsymbol{v}) = \alpha\, f\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) = \alpha \begin{pmatrix} 0 \\ v_1 \\ 2\, v_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha\, v_1 \\ 2\, \alpha\, v_1 \end{pmatrix}$

Comme les résultats sont identiques, on a que $f(\alpha\, \boldsymbol{v}) = \alpha\, f(\boldsymbol{v})$.

L'application f est bien linéaire.