3.8 Commençons par résoudre ce système :

$$\alpha\begin{pmatrix}1\\2\\-6\end{pmatrix}+\beta\begin{pmatrix}-1\\-1\\5\end{pmatrix}+\gamma\begin{pmatrix}0\\-1\\2\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}1&-1&0&x\\2&-1&-1&y\\-6&5&2&z\end{array}\right)\overset{\substack{L_2\to L_2-2L_1\\L_3\to L_3+6L_1}}{\Longrightarrow}\left(\begin{array}{ccc|c}1&-1&0&x\\0&1&-1&-2x+y\\0&-1&2&6x+z\end{array}\right)\overset{L_3\to L_3+L_2}{\Longrightarrow}$$

$$\left(\begin{array}{ccc|c}1&-1&0&x\\0&1&-1&-2x+y\\0&0&1&4x+y+z\end{array}\right)\overset{L_2\to L_2+L_3}{\Longrightarrow}\left(\begin{array}{ccc|c}1&-1&0&x\\0&1&0&2x+2y+z\\0&0&1&4x+y+z\end{array}\right)$$

$$\overset{L_1\to L_1+L_2}{\Longrightarrow}\left(\begin{array}{ccc|c}1&0&0&3x+2y+z\\0&1&0&2x+2y+z\\0&0&1&4x+y+z\end{array}\right)\Longrightarrow\begin{cases}\alpha&=3x+2y+z\\\beta&=2x+2y+z\\\gamma&=4x+y+z\end{cases}$$

Nous pouvons maintenant déterminer l'application linéaire f :

$$f\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right)=f\left((3x+2y+z)\begin{pmatrix}1\\2\\-6\end{pmatrix}+(2x+2y+z)\begin{pmatrix}-1\\-1\\5\end{pmatrix}+(4x+y+z)\begin{pmatrix}5\\3\\-2\end{pmatrix}\right)$$

$$=(3x+2y+z)f\left(\begin{pmatrix}1\\2\\-6\end{pmatrix}\right)+(2x+2y+z)f\left(\begin{pmatrix}-1\\-1\\5\end{pmatrix}\right)+(4x+y+z)f\left(\begin{pmatrix}5\\3\\-2\end{pmatrix}\right)$$

$$=(3x+2y+z)\begin{pmatrix}5\\1\\3\end{pmatrix}+(2x+2y+z)\begin{pmatrix}1\\1\\5\end{pmatrix}+(4x+y+z)\begin{pmatrix}5\\3\\-2\end{pmatrix}$$

$$=\begin{pmatrix}15x+10y+5z\\3x+2y+z\\9x+6y+3z\end{pmatrix}+\begin{pmatrix}2x+2y+z\\2x+2y+z\\10x+10y+5z\end{pmatrix}+\begin{pmatrix}20x+5y+5z\\12x+3y+3z\\-8x-2y-2z\end{pmatrix}$$

$$=\begin{pmatrix}37x+17y+11z\\17x+7y+5z\\11x+14y+6z\end{pmatrix}=\begin{pmatrix}37&17&11\\17&7&5\\11&14&6\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$