6.12 Polynôme caractéristique

$$\begin{vmatrix} 7-\lambda & 2 \\ -4 & 1-\lambda \end{vmatrix} \overset{\mathrm{L}_1 \to \mathrm{L}_1 + \mathrm{L}_2}{=} \begin{vmatrix} 3-\lambda & 3-\lambda \\ -4 & 1-\lambda \end{vmatrix} = (3-\lambda) \begin{vmatrix} 1 & 1 \\ -4 & 1-\lambda \end{vmatrix} \overset{\mathrm{C}_2 \to \mathrm{C}_2 - \mathrm{C}_1}{=}$$

$$(3-\lambda) \begin{vmatrix} 1 & 0 \\ -4 & 5-\lambda \end{vmatrix} = (3-\lambda)\,(5-\lambda)$$

Valeurs propres

$\lambda = 3$ et $\lambda = 5$

Espace propre E_3

$$\left(\begin{array}{cc|c} 7-3 & 2 & 0 \\ -4 & 1-3 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 4 & 2 & 0 \\ -4 & -2 & 0 \end{array}\right) \overset{\substack{\mathrm{L}_1 \to 1/2\,\mathrm{L}_1 \\ \mathrm{L}_2 \to -1/2\,\mathrm{L}_2}}{\Longrightarrow} \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right) \overset{\mathrm{L}_2 \to \mathrm{L}_2 - \mathrm{L}_1}{\Longrightarrow}$$

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \overset{\mathrm{L}_1 \to 1/2\,\mathrm{L}_1}{\Longrightarrow} \left(\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right) \Longrightarrow \begin{cases} x = -\frac{1}{2}\,\alpha \\ y = \phantom{-\frac{1}{2}}\,\alpha \end{cases} = \frac{1}{2}\,\alpha \begin{pmatrix} -1 \\ 2 \end{pmatrix} \text{ où } \alpha \in \mathbb{R}$$

$$\mathrm{E}_3 = \left\{ \alpha \begin{pmatrix} -1 \\ 2 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

Espace propre E_5

$$\left(\begin{array}{cc|c} 7-5 & 2 & 0 \\ -4 & 1-5 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 2 & 2 & 0 \\ -4 & -4 & 0 \end{array}\right) \overset{\substack{\mathrm{L}_1 \to 1/2\,\mathrm{L}_1 \\ \mathrm{L}_2 \to -1/4\,\mathrm{L}_2}}{\Longrightarrow} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right)$$

$$\overset{\mathrm{L}_2 \to \mathrm{L}_2 - \mathrm{L}_1}{\Longrightarrow} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Longrightarrow \begin{cases} x = -\alpha \\ y = \alpha \end{cases} = \alpha \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ où } \alpha \in \mathbb{R}$$

$$\mathrm{E}_5 = \left\{ \alpha \begin{pmatrix} -1 \\ 1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

Diagonalisation

$$\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$$

Inverse de la matrice de passage

$$\left(\begin{array}{cc|cc} -1 & -1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \overset{\mathrm{L}_2 \to \mathrm{L}_2 + 2\,\mathrm{L}_1}{\Longrightarrow} \left(\begin{array}{cc|cc} -1 & -1 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{array}\right) \overset{\substack{\mathrm{L}_1 \to -\mathrm{L}_1 \\ \mathrm{L}_2 \to -\mathrm{L}_2}}{\Longrightarrow}$$

$$\left(\begin{array}{cc|cc} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & -1 \end{array}\right) \overset{\mathrm{L}_1 \to \mathrm{L}_1 - \mathrm{L}_2}{\Longrightarrow} \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 \end{array}\right)$$

Calcul de la puissance

$$\begin{aligned} \mathrm{A}^n &= \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}^n \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3^n & 0 \\ 0 & 5^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -3^n & -5^n \\ 2 \cdot 3^n & -5^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 5^n - 3^n & 5^n - 3^n \\ 2 \cdot 3^n - 2 \cdot 5^n & 2 \cdot 3^n - 5^n \end{pmatrix} \end{aligned}$$