6.13 Polynôme caractéristique

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} \overset{L_1 \to L_1 + L_2}{=} (1-\lambda)\begin{vmatrix} 3-\lambda & 3-\lambda \\ 1 & 2-\lambda \end{vmatrix} =$$

$$(1-\lambda)\,(3-\lambda)\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} \overset{C_2 \to C_2 - C_1}{=} (1-\lambda)\,(3-\lambda)\begin{vmatrix} 1 & 0 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2\,(3-\lambda)$$

Valeurs propres

$\lambda = 1$ et $\lambda = 3$

Espace propre E_1

$$\left(\begin{array}{ccc|c} 2-1 & 0 & 1 & 0 \\ 1 & 1-1 & 1 & 0 \\ 1 & 0 & 2-1 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \overset{\substack{L_2 \to L_2 - L_1 \\ L_3 \to L_3 - L_1}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x = \quad -\beta \\ y = \alpha \\ z = \quad\ \ \beta \end{cases} = \alpha\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ où } \alpha, \beta \in \mathbb{R}$$

$$E_1 = \left\{\alpha\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} : \alpha, \beta \in \mathbb{R}\right\}$$

Espace propre E_3

$$\left(\begin{array}{ccc|c} 2-3 & 0 & 1 & 0 \\ 1 & 1-3 & 1 & 0 \\ 1 & 0 & 2-3 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \overset{\substack{L_2 \to L_2 + L_1 \\ L_3 \to L_3 + L_1}}{\Longrightarrow}$$

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{\substack{L_1 \to -L_1 \\ L_2 \to -1/2\, L_2}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow \begin{cases} x = \alpha \\ y = \alpha \\ z = \alpha \end{cases} = \alpha\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$E_3 = \left\{\alpha\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} : \alpha \in \mathbb{R}\right\}$$

Diagonalisation

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Inverse de la matrice de passage

$$\left(\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \overset{L_1 \leftrightarrow L_2}{\Longrightarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \overset{L_3 \to L_3 + L_2}{\Longrightarrow}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{array}\right) \overset{\substack{L_1 \to 2\,L_1 - L_3 \\ L_2 \to -2\,L_2 + L_3}}{\Longrightarrow} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & -1 & 2 & -1 \\ 0 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{array}\right)$$

$$\xRightarrow[L_3 \to 1/2\, L_3]{\substack{L_1 \to 1/2\, L_1 \\ L_2 \to 1/2\, L_2}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right)$$

Calcul de la puissance

$$\begin{aligned}
\begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}^n &= \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}^n \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3^n \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 & 3^n \\ 1 & 0 & 3^n \\ 0 & 1 & 3^n \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3^n + 1 & 0 & 3^n - 1 \\ 3^n - 1 & 2 & 3^n - 1 \\ 3^n - 1 & 0 & 3^n + 1 \end{pmatrix}
\end{aligned}$$