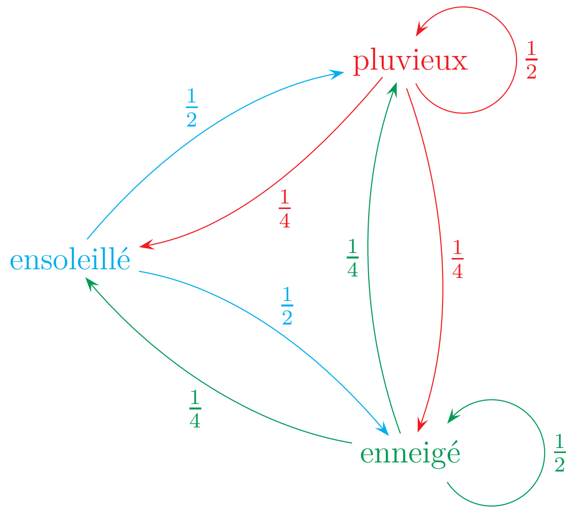
7.7



$$A = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

Calculons les valeurs propres :

$$\det(A - \lambda I) = \begin{vmatrix} 0-\lambda & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2}-\lambda & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2}-\lambda \end{vmatrix} \overset{L_1 \to L_1 + L_2 + L_3}{=} \begin{vmatrix} 1-\lambda & 1-\lambda & 1-\lambda \\ \frac{1}{2} & \frac{1}{2}-\lambda & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2}-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2}-\lambda & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2}-\lambda \end{vmatrix} \overset{\substack{C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1}}{=} (1-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\lambda & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} & -\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} -\lambda & -\frac{1}{4} \\ -\frac{1}{4} & -\lambda \end{vmatrix} \overset{L_2 \to L_2 + L_1}{=} (1-\lambda)\begin{vmatrix} -\lambda & -\frac{1}{4} \\ -\lambda-\frac{1}{4} & -\lambda-\frac{1}{4} \end{vmatrix}$$

$$= (1-\lambda)\,(-\lambda-\tfrac{1}{4})\begin{vmatrix} -\lambda & -\frac{1}{4} \\ 1 & 1 \end{vmatrix} \overset{C_1 \to C_1 - C_2}{=} (1-\lambda)\,(-\lambda-\tfrac{1}{4})\begin{vmatrix} -\lambda+\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 \end{vmatrix}$$

$$= (1-\lambda)\,(-\lambda-\tfrac{1}{4})\,(-\lambda+\tfrac{1}{4})$$

On a obtenu trois valeurs propres : $\lambda = 1$, $\lambda = -\frac{1}{4}$ et $\lambda = \frac{1}{4}$.

Déterminons l'espace propre E_1 :

$$\left(\begin{array}{ccc|c} 0-1 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2}-1 & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2}-1 & 0 \end{array}\right) \overset{\substack{L_1 \to 4L_1 \\ L_2 \to 4L_2 \\ L_3 \to 4L_3}}{\Longrightarrow} \left(\begin{array}{ccc|c} -4 & 1 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right) \overset{L_3 \leftrightarrow L_1}{\Longrightarrow}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 2 & -2 & 1 & 0 \\ -4 & 1 & 1 & 0 \end{array}\right) \overset{\substack{L_2 \to L_2 - L_1 \\ L_3 \to L_3 + 2L_1}}{\Longrightarrow} \left(\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) \overset{\substack{L_2 \to -1/3\, L_2 \\ L_3 \to 1/3\, L_3}}{\Longrightarrow}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \overset{\substack{L_1 \to L_1 - L_2 \\ L_3 \to L_3 - L_2}}{\Longrightarrow} \left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{L_1 \to 1/2\, L_1}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left\{\begin{aligned} x &= \tfrac{1}{2}\,\alpha \\ y &= \phantom{\tfrac{1}{2}}\,\alpha \\ z &= \phantom{\tfrac{1}{2}}\,\alpha \end{aligned}\right. = \tfrac{1}{2}\,\alpha \begin{pmatrix}1\\2\\2\end{pmatrix}$$

Déterminons l'espace propre $\mathrm{E}_{-\frac{1}{4}}$:

$$\left(\begin{array}{ccc|c} 0-(-\frac{1}{4}) & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2}-(-\frac{1}{4}) & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2}-(-\frac{1}{4}) & 0 \end{array}\right) \xRightarrow{\substack{\mathrm{L}_1 \to 4\,\mathrm{L}_1 \\ \mathrm{L}_2 \to 4\,\mathrm{L}_2 \\ \mathrm{L}_3 \to 4\,\mathrm{L}_3}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 0 \\ 2 & 1 & 3 & 0 \end{array}\right) \xRightarrow{\substack{\mathrm{L}_2 \to \mathrm{L}_2 - 2\,\mathrm{L}_1 \\ \mathrm{L}_3 \to \mathrm{L}_3 - 2\,\mathrm{L}_1}}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \xRightarrow{\substack{\mathrm{L}_1 \to \mathrm{L}_1 - \mathrm{L}_2 \\ \mathrm{L}_3 \to \mathrm{L}_3 + \mathrm{L}_2}} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow \left\{\begin{aligned} x &= -2\,\alpha \\ y &= \,\alpha \\ z &= \,\alpha \end{aligned}\right. = \alpha \begin{pmatrix}-2\\1\\1\end{pmatrix}$$

Déterminons l'espace propre $\mathrm{E}_{\frac{1}{4}}$:

$$\left(\begin{array}{ccc|c} 0-\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2}-\frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2}-\frac{1}{4} & 0 \end{array}\right) \xRightarrow{\substack{\mathrm{L}_1 \to -4\,\mathrm{L}_1 \\ \mathrm{L}_2 \to 4\,\mathrm{L}_2 \\ \mathrm{L}_3 \to 4\,\mathrm{L}_3}} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right) \xRightarrow{\substack{\mathrm{L}_2 \to \mathrm{L}_2 - 2\,\mathrm{L}_1 \\ \mathrm{L}_3 \to \mathrm{L}_3 - 2\,\mathrm{L}_1}}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right) \xRightarrow{\substack{\mathrm{L}_2 \to 1/3\,\mathrm{L}_2 \\ \mathrm{L}_3 \to 1/3\,\mathrm{L}_3}} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xRightarrow{\substack{\mathrm{L}_1 \to \mathrm{L}_1 + \mathrm{L}_2 \\ \mathrm{L}_3 \to \mathrm{L}_3 - \mathrm{L}_2}}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow \left\{\begin{aligned} x &= 0 \\ y &= -\alpha \\ z &= \alpha \end{aligned}\right. = \alpha \begin{pmatrix}0\\-1\\1\end{pmatrix}$$

Calculons l'inverse de la matrice de passage $\mathrm{P} = \begin{pmatrix}1 & -2 & 0 \\ 2 & 1 & -1 \\ 2 & 1 & 1\end{pmatrix}$:

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xRightarrow{\substack{\mathrm{L}_2 \to \mathrm{L}_2 - 2\,\mathrm{L}_1 \\ \mathrm{L}_3 \to \mathrm{L}_3 - 2\,\mathrm{L}_1}} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 5 & -1 & -2 & 1 & 0 \\ 0 & 5 & 1 & -2 & 0 & 1 \end{array}\right) \xRightarrow{\mathrm{L}_3 \to \mathrm{L}_3 - \mathrm{L}_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 5 & -1 & -2 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right) \xRightarrow{\mathrm{L}_2 \to 2\,\mathrm{L}_2 + \mathrm{L}_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 10 & 0 & -4 & 1 & 1 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right) \xRightarrow{\mathrm{L}_1 \to 5\,\mathrm{L}_1 + \mathrm{L}_2}$$

$$\left(\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 1 & 1 \\ 0 & 10 & 0 & -4 & 1 & 1 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right) \xRightarrow{\substack{\mathrm{L}_1 \to 1/5\,\mathrm{L}_1 \\ \mathrm{L}_2 \to 1/10\,\mathrm{L}_2 \\ \mathrm{L}_3 \to 1/2\,\mathrm{L}_3}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & 0 & -\frac{2}{5} & \frac{1}{10} & \frac{1}{10} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right)$$

Calculons A^n :

$$\mathrm{A}^n = \begin{pmatrix}1 & -2 & 0 \\ 2 & 1 & -1 \\ 2 & 1 & 1\end{pmatrix} \begin{pmatrix}1 & 0 & 0 \\ 0 & -(\frac{1}{4})^n & 0 \\ 0 & 0 & \frac{1}{4}^n\end{pmatrix} \begin{pmatrix}\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{10} & \frac{1}{10} \\ 0 & -\frac{1}{2} & \frac{1}{2}\end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2\left(-\frac{1}{4}\right)^n & 0 \\ 2 & \left(-\frac{1}{4}\right)^n & -\left(\frac{1}{4}\right)^n \\ 2 & \left(-\frac{1}{4}\right)^n & \left(\frac{1}{4}\right)^n \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{10} & \frac{1}{10} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} + \frac{4}{5}\left(-\frac{1}{4}\right)^n & \frac{1}{5} - \frac{1}{5}\left(-\frac{1}{4}\right)^n & \frac{1}{5} - \frac{1}{5}\left(-\frac{1}{4}\right)^n \\ \frac{2}{5} - \frac{2}{5}\left(-\frac{1}{4}\right)^n & \frac{2}{5} + \frac{1}{2}\left(\frac{1}{4}\right)^n + \frac{1}{10}\left(-\frac{1}{4}\right)^n & \frac{2}{5} - \frac{1}{2}\left(\frac{1}{4}\right)^n + \frac{1}{10}\left(-\frac{1}{4}\right)^n \\ \frac{2}{5} - \frac{2}{5}\left(-\frac{1}{4}\right)^n & \frac{2}{5} - \frac{1}{2}\left(\frac{1}{4}\right)^n + \frac{1}{10}\left(-\frac{1}{4}\right)^n & \frac{2}{5} + \frac{1}{2}\left(\frac{1}{4}\right)^n + \frac{1}{10}\left(-\frac{1}{4}\right)^n \end{pmatrix}$$

Pour une prédiction à long terme, on effectue le passage à la limite :

$$\lim_{n \to +\infty} \mathrm{A}^n = \lim_{n \to +\infty} \begin{pmatrix} \frac{1}{5} + \frac{4}{5}\left(-\frac{1}{4}\right)^n & \frac{1}{5} - \frac{1}{5}\left(-\frac{1}{4}\right)^n & \frac{1}{5} - \frac{1}{5}\left(-\frac{1}{4}\right)^n \\ \frac{2}{5} - \frac{2}{5}\left(-\frac{1}{4}\right)^n & \frac{2}{5} + \frac{1}{2}\left(\frac{1}{4}\right)^n + \frac{1}{10}\left(-\frac{1}{4}\right)^n & \frac{2}{5} - \frac{1}{2}\left(\frac{1}{4}\right)^n + \frac{1}{10}\left(-\frac{1}{4}\right)^n \\ \frac{2}{5} - \frac{2}{5}\left(-\frac{1}{4}\right)^n & \frac{2}{5} - \frac{1}{2}\left(\frac{1}{4}\right)^n + \frac{1}{10}\left(-\frac{1}{4}\right)^n & \frac{2}{5} + \frac{1}{2}\left(\frac{1}{4}\right)^n + \frac{1}{10}\left(-\frac{1}{4}\right)^n \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix} \qquad \text{soleil : } \frac{1}{5}, \text{ pluie : } \frac{2}{5}, \text{ neige : } \frac{2}{5}$$