7.2 $f(x) = \sin(x)$

$f'(x) = \big(\sin(x)\big)' = \cos(x)$

$f''(x) = \big(\cos(x)\big)' = -\sin(x)$

$f^{(3)}(x) = \big(-\sin(x)\big)' = -\cos(x)$

$f^{(4)}(x) = \big(-\cos(x)\big)' = \sin(x) = f(x)$

Plus généralement, on obtient :

$$f^{(k)}(x) = \begin{cases} \cos(x) & \text{si } k \equiv 1 \mod 4 \\ -\sin(x) & \text{si } k \equiv 2 \mod 4 \\ -\cos(x) & \text{si } k \equiv 3 \mod 4 \\ \sin(x) & \text{si } k \equiv 0 \mod 4 \end{cases}$$

Il en résulte :

$$f^{(k)}(0) = \begin{cases} 1 & \text{si } k \equiv 1 \mod 4 \\ 0 & \text{si } k \equiv 2 \mod 4 \\ -1 & \text{si } k \equiv 3 \mod 4 \\ 0 & \text{si } k \equiv 0 \mod 4 \end{cases}$$

1) $$\begin{aligned} \mathrm{P}_1(x) &= f(a) + f'(a)\,(x-a) \\ &= 0 + 1\,(x-0) \\ &= x \end{aligned}$$

2) $$\begin{aligned} \mathrm{P}_2(x) &= f(a) + f'(a)\,(x-a) + \tfrac{f''(a)}{2!}\,(x-a)^2 \\ &= 0 + 1\,(x-0) + \tfrac{0}{2!}\,(x-0)^2 \\ &= x \end{aligned}$$

3) $$\begin{aligned} \mathrm{P}_3(x) &= f(a) + f'(a)\,(x-a) + \tfrac{f''(a)}{2!}\,(x-a)^2 + \tfrac{f^{(3)}(a)}{3!}\,(x-a)^3 \\ &= 0 + 1\,(x-0) + \tfrac{0}{2!}\,(x-0)^2 + \tfrac{-1}{3!}\,(x-0)^3 \\ &= x - \tfrac{x^3}{3!} \end{aligned}$$

4) $$\begin{aligned} \mathrm{P}_4(x) &= f(a) + f'(a)\,(x-a) + \tfrac{f''(a)}{2!}\,(x-a)^2 + \tfrac{f^{(3)}(a)}{3!}\,(x-a)^3 + \tfrac{f^{(4)}(a)}{4!}\,(x-a)^4 \\ &= 0 + 1\,(x-0) + \tfrac{0}{2!}\,(x-0)^2 + \tfrac{-1}{3!}\,(x-0)^3 + \tfrac{0}{4!}\,(x-0)^4 \\ &= x - \tfrac{x^3}{3!} \end{aligned}$$

5) $$\begin{aligned} \mathrm{P}_5(x) &= f(a) + f'(a)\,(x-a) + \tfrac{f''(a)}{2!}\,(x-a)^2 + \tfrac{f^{(3)}(a)}{3!}\,(x-a)^3 \\ &\quad + \tfrac{f^{(4)}(a)}{4!}\,(x-a)^4 + \tfrac{f^{(5)}(a)}{5!}\,(x-a)^5 \\ &= 0 + 1\,(x-0) + \tfrac{0}{2!}\,(x-0)^2 + \tfrac{-1}{3!}\,(x-0)^3 \\ &\quad + \tfrac{0}{4!}\,(x-0)^4 + \tfrac{1}{5!}\,(x-0)^5 \\ &= x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} \end{aligned}$$

6) $$\begin{aligned} P_7(x) &= f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2 + \frac{f^{(3)}(a)}{3!}\,(x-a)^3 \\ &+ \frac{f^{(4)}(a)}{4!}\,(x-a)^4 + \frac{f^{(5)}(a)}{5!}\,(x-a)^5 + \frac{f^{(6)}(a)}{6!}\,(x-a)^6 + \frac{f^{(7)}(a)}{7!}\,(x-a)^7 \\ &= 0 + 1\,(x-0) + \frac{0}{2!}\,(x-0)^2 + \frac{-1}{3!}\,(x-0)^3 \\ &+ \frac{0}{4!}\,(x-0)^4 + \frac{1}{5!}\,(x-0)^5 + \frac{0}{6!}\,(x-0)^6 + \frac{-1}{7!}\,(x-0)^7 \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \end{aligned}$$

7) On généralise facilement le résultat :
$$P_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \ldots + (-1)^n\,\frac{x^{2n+1}}{(2\,n+1)!}$$