7.22 1) $\dfrac{\pi}{4} = \arctan(1) = 1 - \dfrac{1^3}{3} + \dfrac{1^5}{5} - \dfrac{1^7}{7} + \ldots + (-1)^k\,\dfrac{1^{2k+1}}{2\,k+1} + \ldots$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + (-1)^k\,\frac{1}{2\,k+1} + \ldots$$

En quadruplant cette égalité, on obtient la formule :
$\pi = 4\left(1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots + (-1)^k\,\dfrac{1}{2\,k+1} + \ldots\right)$

3) (a) $\tan(\alpha+\beta+\gamma) = \tan\big(\alpha+(\beta+\gamma)\big) = \dfrac{\tan(\alpha)+\tan(\beta+\gamma)}{1-\tan(\alpha)\,\tan(\beta+\gamma)}$

$$= \frac{\tan(\alpha) + \dfrac{\tan(\beta)+\tan(\gamma)}{1-\tan(\beta)\,\tan(\gamma)}}{1-\tan(\alpha)\,\dfrac{\tan(\beta)+\tan(\gamma)}{1-\tan(\beta)\,\tan(\gamma)}}$$

$$= \frac{\dfrac{\tan(\alpha)\,\big(1-\tan(\beta)\,\tan(\gamma)\big)+\big(\tan(\beta)+\tan(\gamma)\big)}{1-\tan(\beta)\,\tan(\gamma)}}{\dfrac{1\,\big(1-\tan(\beta)\,\tan(\gamma)\big)-\tan(\alpha)\big(\tan(\beta)+\tan(\gamma)\big)}{1-\tan(\beta)\,\tan(\gamma)}}$$

$$= \frac{\tan(\alpha)-\tan(\alpha)\,\tan(\beta)\,\tan(\gamma)+\tan(\beta)+\tan(\gamma)}{1-\tan(\beta)\,\tan(\gamma)-\tan(\alpha)\,\tan(\beta)-\tan(\alpha)\,\tan(\gamma)}$$

$$= \frac{\tan(\alpha)+\tan(\beta)+\tan(\gamma)-\tan(\alpha)\,\tan(\beta)\,\tan(\gamma)}{1-\tan(\alpha)\,\tan(\beta)-\tan(\alpha)\,\tan(\gamma)-\tan(\beta)\,\tan(\gamma)}$$

(b) $\tan\Big(\arctan\big(\tfrac{1}{2}\big)+\arctan\big(\tfrac{1}{5}\big)+\arctan\big(\tfrac{1}{8}\big)\Big) =$

$$\frac{\tan(\arctan(\frac{1}{2}))+\tan(\arctan(\frac{1}{5}))+\tan(\arctan(\frac{1}{8}))-\tan(\arctan(\frac{1}{2}))\,\tan(\arctan(\frac{1}{5}))\,\tan(\arctan(\frac{1}{8}))}{1-\tan(\arctan(\frac{1}{2}))\,\tan(\arctan(\frac{1}{5}))-\tan(\arctan(\frac{1}{2}))\,\tan(\arctan(\frac{1}{8}))-\tan(\arctan(\frac{1}{5}))\,\tan(\arctan(\frac{1}{8}))} =$$

$$\frac{\frac{1}{2}+\frac{1}{5}+\frac{1}{8}-\frac{1}{2}\cdot\frac{1}{5}\cdot\frac{1}{8}}{1-\frac{1}{2}\cdot\frac{1}{5}-\frac{1}{2}\cdot\frac{1}{8}-\frac{1}{5}\cdot\frac{1}{8}} = \frac{\frac{1}{2}+\frac{1}{5}+\frac{1}{8}-\frac{1}{80}}{1-\frac{1}{10}-\frac{1}{16}-\frac{1}{40}} = \frac{\frac{40}{80}+\frac{16}{80}+\frac{10}{80}-\frac{1}{80}}{\frac{80}{80}-\frac{8}{80}-\frac{5}{80}-\frac{2}{80}} =$$

$$\frac{\frac{65}{80}}{\frac{65}{80}} = 1 = \tan\big(\tfrac{\pi}{4}\big)$$

On en tire que $\arctan\big(\frac{1}{2}\big)+\arctan\big(\frac{1}{5}\big)+\arctan\big(\frac{1}{8}\big) = \frac{\pi}{4}+k\,\pi$ où $k\in\mathbb{Z}$.

$8>5>2=\sqrt{4}>\sqrt{3}$ implique $\frac{1}{8}<\frac{1}{5}<\frac{1}{2}<\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$

Étant donné que la fonction $\arctan(x)$ est croissante, il en résulte :
$0=\arctan(0)<\arctan\big(\frac{1}{8}\big)<\arctan\big(\frac{1}{5}\big)<\arctan\big(\frac{1}{2}\big)<\arctan\big(\frac{\sqrt{3}}{3}\big)=\frac{\pi}{6}$
Donc $0<\arctan\big(\frac{1}{2}\big)+\arctan\big(\frac{1}{5}\big)+\arctan\big(\frac{1}{8}\big)<\frac{\pi}{6}+\frac{\pi}{6}+\frac{\pi}{6}=\frac{\pi}{2}$
C'est pourquoi, on a bien $\arctan\big(\frac{1}{2}\big)+\arctan\big(\frac{1}{5}\big)+\arctan\big(\frac{1}{8}\big)=\frac{\pi}{4}$.