7.6

1) $f(x) = \sin(x)$ $\qquad f(a) = f\big(\frac{\pi}{2}\big) = \sin\big(\frac{\pi}{2}\big) = 1$

$f'(x) = \cos(x)$ $\qquad f'(a) = f'\big(\frac{\pi}{2}\big) = \cos\big(\frac{\pi}{2}\big) = 0$

$f''(x) = -\sin(x)$ $\qquad f''(a) = f''\big(\frac{\pi}{2}\big) = -\sin\big(\frac{\pi}{2}\big) = -1$

$f^{(3)}(x) = -\cos(x)$ $\qquad f^{(3)}(a) = f^{(3)}\big(\frac{\pi}{2}\big) = -\cos\big(\frac{\pi}{2}\big) = 0$

$f^{(4)}(x) = \sin(x)$ $\qquad f^{(4)}(a) = f^{(4)}\big(\frac{\pi}{2}\big) = \sin\big(\frac{\pi}{2}\big) = 1$

$f^{(5)}(x) = \cos(x)$ $\qquad f^{(5)}(a) = f^{(5)}\big(\frac{\pi}{2}\big) = \cos\big(\frac{\pi}{2}\big) = 0$

$$\begin{aligned}\mathrm{P}_5(x) &= f(a) + f'(a)\,(x-a) + \tfrac{f''(a)}{2!}\,(x-a)^2 + \tfrac{f^{(3)}(a)}{3!}\,(x-a)^3 + \tfrac{f^{(4)}(a)}{4!}\,(x-a)^4 + \tfrac{f^{(5)}(a)}{5!}\,(x-a)^5\\ &= 1 + 0\,(x-\tfrac{\pi}{2}) + \tfrac{-1}{2!}\,(x-\tfrac{\pi}{2})^2 + \tfrac{0}{3!}\,(x-\tfrac{\pi}{2})^3 + \tfrac{1}{4!}\,(x-\tfrac{\pi}{2})^4 + \tfrac{0}{5!}\,(x-\tfrac{\pi}{2})^5\\ &= 1 - \tfrac{1}{2}\left(x-\tfrac{\pi}{2}\right)^2 + \tfrac{1}{4!}\left(x-\tfrac{\pi}{2}\right)^4\end{aligned}$$

2) $$\begin{aligned}\sin(100^\circ) = \sin(\tfrac{5}{9}\cdot 180^\circ) = \sin(\tfrac{5\pi}{9}) &\approx 1 - \tfrac{1}{2}\left(\tfrac{5\pi}{9}-\tfrac{\pi}{2}\right)^2 + \tfrac{1}{4!}\left(\tfrac{5\pi}{9}-\tfrac{\pi}{2}\right)^4\\ &\approx 1 - \tfrac{1}{2}\left(\tfrac{\pi}{18}\right)^2 + \tfrac{1}{24}\left(\tfrac{\pi}{18}\right)^4 = \tfrac{2\ 519\ 424-3888\,\pi^2+\pi^4}{2\ 519\ 424} \approx 0{,}984\ 808\end{aligned}$$

3) $$\mathrm{R}_5\big(\tfrac{5\pi}{9}\big) = \frac{f^{(5+1)}(c)}{(5+1)!}\,(\tfrac{5\pi}{9}-\tfrac{\pi}{2})^{5+1} = \frac{-\sin(c)}{6!}\cdot\left(\tfrac{\pi}{18}\right)^6 \quad \text{où } c \in [\tfrac{\pi}{2}\,;\tfrac{5\pi}{9}]$$

Comme $\big|-\sin(c)\big| \leqslant 1$ pour tout $c \in [\frac{\pi}{2}\,;\frac{5\pi}{9}]$, il s'ensuit que :

$\left|\mathrm{R}_5\big(\frac{5\pi}{9}\big)\right| \leqslant \frac{1}{6!}\left(\frac{\pi}{18}\right)^6 \approx 3{,}926\cdot 10^{-8}$