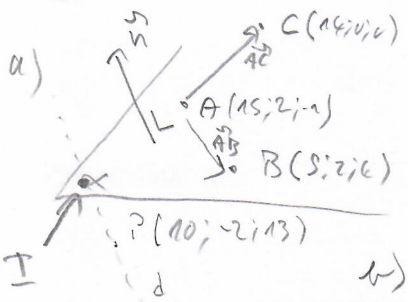
MR - 2019 - Chamblandes

(P1) à (P4) : cf MS (et (P6))

(P5) a)



$\vec{AB} = \begin{pmatrix}-10\\0\\5\end{pmatrix} = -5\begin{pmatrix}2\\0\\-1\end{pmatrix}$ $\quad \vec{AC} = \begin{pmatrix}-1\\-2\\1\end{pmatrix}$

$\vec{n} = \begin{pmatrix}2\\0\\-1\end{pmatrix} \times \begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{vmatrix}\vec{e}_1 & 2 & 1\\ \vec{e}_2 & 0 & 2\\ \vec{e}_3 & -1 & -1\end{vmatrix} = \begin{pmatrix}2\\1\\4\end{pmatrix} \Rightarrow \alpha: 2x+1y+4z \overset{C}{=} 2\cdot 14+0+0=28$

$\alpha: 2x+y+4z=28$

b) $d: \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}10\\-2\\13\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\4\end{pmatrix}$

c) $I = d \cap \alpha: \ 2(10+2\lambda)+(-2+\lambda)+4(13+4\lambda)=28 \Leftrightarrow 20+4\lambda-2+\lambda+52+16\lambda=28$

$\Leftrightarrow 21\lambda=-42 \Leftrightarrow \lambda=-2 \Rightarrow I(10-4; -2-2; 13-8) = (6;-4;5)$

d) $r = \|\vec{PI}\| = \left\|\begin{pmatrix}6-10\\-4+2\\5-13\end{pmatrix}\right\| = \left\|\begin{pmatrix}-4\\-2\\-8\end{pmatrix}\right\| = 2\sqrt{4+1+16} = 2\sqrt{21} \Rightarrow r^2 = 4\cdot 21 = 84$

ainsi $\Sigma_1: (x-10)^2+(y+2)^2+(z-13)^2=84$

e) $M_{IP} = \left(\frac{10+6}{2}; \frac{-2-4}{2}; \frac{13+5}{2}\right) = (8;-3;9) = C_2$ et $r = \frac{2\sqrt{21}}{2} = \sqrt{21} \Rightarrow \Sigma_2: (x-8)^2+(y+3)^2+(z-9)^2=21$

(P7) a) $|A_k| = \begin{vmatrix}-4 & 2 & 2\\ 6 & k & -2\\ -18 & 6 & k+8\end{vmatrix} \underset{C_2\to C_2-C_3}{\overset{C_1\to C_1+2C_3}{=}} \begin{vmatrix}0 & 0 & 2\\ 2 & k+2 & -2\\ 2k-2 & -k-2 & k+8\end{vmatrix} = 2(k+2)\begin{vmatrix}2 & 1\\ 2k-2 & -1\end{vmatrix} = 2(k+2)(-2-2k+2) = 2(k+2)(-2k)$

ainsi A_k n'est plus inversible si $\alpha \in \{0;-2\}$.

b) $A = A_0 = \begin{pmatrix}-4 & 2 & 2\\ 6 & 0 & -2\\ -18 & 6 & 8\end{pmatrix} \underset{(-1)}{\overset{:2}{\Leftrightarrow}} \begin{pmatrix}2 & -1 & -1\\ 3 & 0 & -1\\ -9 & 3 & 4\end{pmatrix} \underset{L_3\to 2L_3+9L_1}{\overset{L_2\to 2L_2-3L_1}{\Leftrightarrow}} \begin{pmatrix}2 & -1 & -1\\ 0 & 3 & 1\\ 0 & -3 & -1\end{pmatrix} \Leftrightarrow \begin{pmatrix}2 & -1 & -1\\ 0 & 3 & 1\\ 0 & 0 & 0\end{pmatrix} \underset{L_2\to \frac{L_2}{3}}{\overset{L_1\to L_1+\frac{L_2}{3}}{\Leftrightarrow}} \begin{pmatrix}2 & 0 & -2/3\\ 0 & 1 & 1/3\\ 0 & 0 & 0\end{pmatrix} \Leftrightarrow \begin{pmatrix}1 & 0 & -1/3\\ 0 & 1 & 1/3\\ 0 & 0 & 0\end{pmatrix}$

donc base $Ker(A) = \left(\begin{pmatrix}1\\-1\\3\end{pmatrix}\right)$ et base $Im(A) = \left(\begin{pmatrix}-4\\6\\-18\end{pmatrix}; \begin{pmatrix}2\\0\\6\end{pmatrix}\right)$

c) $|A-\lambda I_3| = \begin{vmatrix}-4-\lambda & 2 & 2\\ 6 & -\lambda & -2\\ -18 & 6 & 8-\lambda\end{vmatrix} \overset{L_2\to L_2+L_1}{=} \begin{vmatrix}-4-\lambda & 2 & 2\\ 2-\lambda & 2-\lambda & 0\\ -18 & 6 & 8-\lambda\end{vmatrix} \overset{C_2\to C_2-C_1}{=} (2-\lambda)\begin{vmatrix}-4-\lambda & \lambda+6 & 2\\ 1 & 0 & 0\\ -18 & 24 & 8-\lambda\end{vmatrix} = (\lambda-2)\begin{vmatrix}\lambda+6 & 2\\ 24 & 8-\lambda\end{vmatrix} = (\lambda-2)(8\lambda+\cancel{48}-\lambda^2-6\lambda\cancel{-48}) =$

$= (\lambda-2)(2\lambda-\lambda^2) = (\lambda-2)\lambda(2-\lambda) = -\lambda(\lambda-2)^2$

On a déjà calculé $E_0 = \left\langle\begin{pmatrix}1\\-1\\3\end{pmatrix}\right\rangle$ $(= Ker(A_0))$

Calculons E_2: $\begin{pmatrix}-6 & 2 & 2\\ 6 & -2 & -2\\ -18 & 6 & 6\end{pmatrix} \Rightarrow \begin{pmatrix}3 & -1 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ donc $E_2 = \left\langle\begin{pmatrix}3\\1\\0\end{pmatrix}; \begin{pmatrix}3\\0\\1\end{pmatrix}\right\rangle$

donc si on choisit $P = \begin{pmatrix}1 & 3 & 3\\ -1 & 1 & 0\\ 3 & 0 & 1\end{pmatrix}$ on aura $D = \begin{pmatrix}0 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix}$

d) Si v est un vecteur propre de A de valeur propre λ: $Av = \lambda v$ alors

$A^4 v = A^3(Av) = A^3(\lambda v) = \lambda A^3 v = \lambda A^2(Av) = \lambda A^2(\lambda v) = \lambda^2 A^2 v = \lambda^2 A(Av) = \lambda^2 A\lambda v = \lambda^3 Av = \lambda^3\lambda v = \lambda^4 v$

ainsi les valeurs propres deviennent λ^4 et les espaces propres restent les mêmes (mais associés à λ^4).

Comme $0^4=0$, $\lambda=0$ et $E_0 = \left\langle\begin{pmatrix}1\\-1\\3\end{pmatrix}\right\rangle$

Par contre $2^4=16$ donc $E_{16} = \left\langle\begin{pmatrix}3\\1\\0\end{pmatrix}; \begin{pmatrix}3\\0\\1\end{pmatrix}\right\rangle$