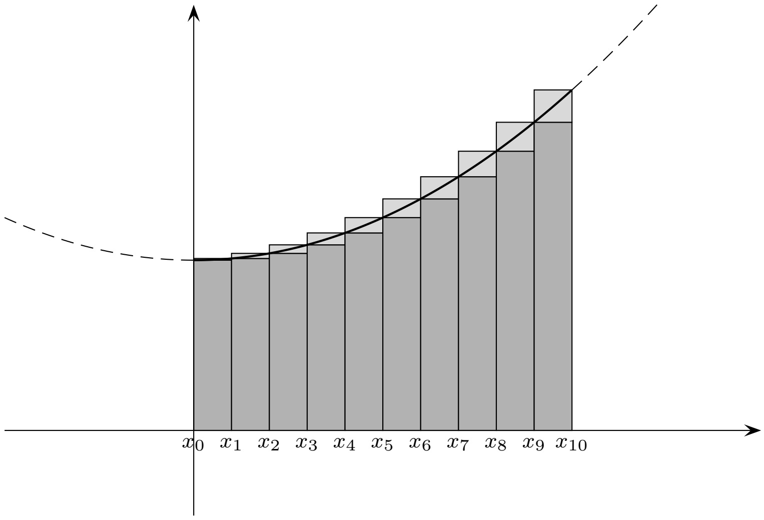
11.1



1) $$\begin{aligned} a_{10} &= \tfrac{1}{10}\,f(x_0)+\tfrac{1}{10}\,f(x_1)+\tfrac{1}{10}\,f(x_2)+\tfrac{1}{10}\,f(x_3)+\tfrac{1}{10}\,f(x_4)+\tfrac{1}{10}\,f(x_5)\\ &\quad+\tfrac{1}{10}\,f(x_6)+\tfrac{1}{10}\,f(x_7)+\tfrac{1}{10}\,f(x_8)+\tfrac{1}{10}\,f(x_9)\\ &= \tfrac{1}{10}\cdot(0^2+1)+\tfrac{1}{10}\cdot\big((\tfrac{1}{10})^2+1\big)+\tfrac{1}{10}\cdot\big((\tfrac{2}{10})^2+1\big)+\tfrac{1}{10}\cdot\big((\tfrac{3}{10})^2+1\big)\\ &\quad+\tfrac{1}{10}\cdot\big((\tfrac{4}{10})^2+1\big)+\tfrac{1}{10}\cdot\big((\tfrac{5}{10})^2+1\big)+\tfrac{1}{10}\cdot\big((\tfrac{6}{10})^2+1\big)+\tfrac{1}{10}\cdot\big((\tfrac{7}{10})^2+1\big)\\ &\quad+\tfrac{1}{10}\cdot\big((\tfrac{8}{10})^2+1\big)+\tfrac{1}{10}\cdot\big((\tfrac{9}{10})^2+1\big)\\ &= \tfrac{1}{10}\,(0+1)+\tfrac{1}{10}\,(\tfrac{1}{100}+1)+\tfrac{1}{10}\,(\tfrac{4}{100}+1)+\tfrac{1}{10}\,(\tfrac{9}{100}+1)+\tfrac{1}{10}\,(\tfrac{16}{100}+1)\\ &\quad+\tfrac{1}{10}\,(\tfrac{25}{100}+1)+\tfrac{1}{10}\,(\tfrac{36}{100}+1)+\tfrac{1}{10}\,(\tfrac{49}{100}+1)+\tfrac{1}{10}\,(\tfrac{64}{100}+1)+\tfrac{1}{10}\,(\tfrac{81}{100}+1)\\ &= \tfrac{1}{10}\cdot 1+\tfrac{1}{10}\cdot\tfrac{101}{100}+\tfrac{1}{10}\cdot\tfrac{104}{100}+\tfrac{1}{10}\cdot\tfrac{109}{100}+\tfrac{1}{10}\cdot\tfrac{116}{100}+\tfrac{1}{10}\cdot\tfrac{125}{100}+\tfrac{1}{10}\cdot\tfrac{136}{100}\\ &\quad+\tfrac{1}{10}\cdot\tfrac{149}{100}+\tfrac{1}{10}\cdot\tfrac{164}{100}+\tfrac{1}{10}\cdot\tfrac{181}{100}\\ &= \tfrac{100}{1000}+\tfrac{101}{1000}+\tfrac{104}{1000}+\tfrac{109}{1000}+\tfrac{116}{1000}+\tfrac{125}{1000}+\tfrac{136}{1000}+\tfrac{149}{1000}+\tfrac{164}{1000}+\tfrac{181}{1000}\\ &= \tfrac{1285}{1000} \end{aligned}$$

$$\begin{aligned} \mathrm{A}_{10} &= \tfrac{1}{10}\,f(x_1)+\tfrac{1}{10}\,f(x_2)+\tfrac{1}{10}\,f(x_3)+\tfrac{1}{10}\,f(x_4)+\tfrac{1}{10}\,f(x_5)+\tfrac{1}{10}\,f(x_6)\\ &\quad+\tfrac{1}{10}\,f(x_7)+\tfrac{1}{10}\,f(x_8)+\tfrac{1}{10}\,f(x_9)+\tfrac{1}{10}\,f(x_{10})\\ &= \tfrac{1}{10}\cdot\big((\tfrac{1}{10})^2+1\big)+\tfrac{1}{10}\cdot\big((\tfrac{2}{10})^2+1\big)+\tfrac{1}{10}\cdot\big((\tfrac{3}{10})^2+1\big)\tfrac{1}{10}\cdot\big((\tfrac{4}{10})^2+1\big)\\ &\quad+\tfrac{1}{10}\cdot\big((\tfrac{5}{10})^2+1\big)+\tfrac{1}{10}\cdot\big((\tfrac{6}{10})^2+1\big)+\tfrac{1}{10}\cdot\big((\tfrac{7}{10})^2+1\big)+\tfrac{1}{10}\cdot\big((\tfrac{8}{10})^2+1\big)\\ &\quad+\tfrac{1}{10}\cdot\big((\tfrac{9}{10})^2+1\big)+\tfrac{1}{10}\cdot(1^2+1)\\ &= \tfrac{1}{10}\,(\tfrac{1}{100}+1)+\tfrac{1}{10}\,(\tfrac{4}{100}+1)+\tfrac{1}{10}\,(\tfrac{9}{100}+1)+\tfrac{1}{10}\,(\tfrac{16}{100}+1)+\tfrac{1}{10}\,(\tfrac{25}{100}+1)\\ &\quad+\tfrac{1}{10}\,(\tfrac{36}{100}+1)+\tfrac{1}{10}\,(\tfrac{49}{100}+1)+\tfrac{1}{10}\,(\tfrac{64}{100}+1)+\tfrac{1}{10}\,(\tfrac{81}{100}+1)+\tfrac{1}{10}\,(1+1)\\ &= \tfrac{1}{10}\cdot\tfrac{101}{100}+\tfrac{1}{10}\cdot\tfrac{104}{100}+\tfrac{1}{10}\cdot\tfrac{109}{100}+\tfrac{1}{10}\cdot\tfrac{116}{100}+\tfrac{1}{10}\cdot\tfrac{125}{100}+\tfrac{1}{10}\cdot\tfrac{136}{100}+\tfrac{1}{10}\cdot\tfrac{149}{100}\\ &\quad+\tfrac{1}{10}\cdot\tfrac{164}{100}+\tfrac{1}{10}\cdot\tfrac{181}{100}+\tfrac{1}{10}\cdot 2\\ &= \tfrac{100}{1000}+\tfrac{101}{1000}+\tfrac{104}{1000}+\tfrac{109}{1000}+\tfrac{116}{1000}+\tfrac{125}{1000}+\tfrac{136}{1000}+\tfrac{149}{1000}+\tfrac{164}{1000}+\tfrac{181}{1000}+\tfrac{200}{1000}\\ &= \tfrac{1385}{1000} \end{aligned}$$

2) $a_n = \sum_{i=0}^{n-1} f(x_i)\,dx_i = \sum_{i=0}^{n-1}(x_i^2+1)\,(x_{i+1}-x_i) = \sum_{i=0}^{n-1}\big((\tfrac{i}{n})^2+1\big)\cdot\tfrac{1}{n}$

$= \tfrac{1}{n}\sum_{i=0}^{n-1}\big((\tfrac{i}{n})^2+1\big) = \tfrac{1}{n}\sum_{i=0}^{n-1}(\tfrac{i}{n})^2 + \tfrac{1}{n}\sum_{i=0}^{n-1}1 = \tfrac{1}{n}\sum_{i=0}^{n-1}\tfrac{1}{n^2}\cdot i^2 + \tfrac{1}{n}\sum_{i=0}^{n-1}1$

$= \tfrac{1}{n^3}\sum_{i=0}^{n-1} i^2 + \tfrac{1}{n}\sum_{i=0}^{n-1}1 = \tfrac{1}{n^3}\cdot\frac{(n-1)\,n\,\big(2\,(n-1)+1\big)}{6} + \tfrac{1}{n}\cdot n = \frac{(n-1)\,n\,(2\,n-1)}{6\,n^3}+1$

$\mathrm{A}_n = \sum_{i=0}^{n-1} f(x_{i+1})\,dx_i = \sum_{i=0}^{n-1}(x_{i+1}^2+1)\,(x_{i+1}-x_i) = \sum_{i=0}^{n-1}\big((\tfrac{i+1}{n})^2+1\big)\cdot\tfrac{1}{n}$

$= \sum_{i=1}^{n}\big((\tfrac{i}{n})^2+1\big)\cdot\tfrac{1}{n} = \tfrac{1}{n}\sum_{i=1}^{n}\big((\tfrac{i}{n})^2+1\big) = \tfrac{1}{n}\sum_{i=1}^{n}(\tfrac{i}{n})^2 + \tfrac{1}{n}\sum_{i=1}^{n}1$

$= \tfrac{1}{n}\sum_{i=1}^{n}\tfrac{1}{n^2}\cdot i^2 + \tfrac{1}{n}\sum_{i=1}^{n}1 = \tfrac{1}{n^3}\sum_{i=1}^{n} i^2 + \tfrac{1}{n}\sum_{i=1}^{n}1 = \tfrac{1}{n^3}\cdot\frac{n\,(n+1)\,(2\,n+1)}{6} + \tfrac{1}{n}\cdot n$

$= \frac{n\,(n+1)\,(2\,n+1)}{6\,n^3}+1$

3) $\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty}\big(\frac{(n-1)\,n\,(2\,n-1)}{6\,n^3}+1\big) = \big(\lim\limits_{n\to+\infty}\frac{(n-1)\,n\,(2\,n-1)}{6\,n^3}\big)+1$

$= \big(\lim\limits_{n\to+\infty}\frac{n\cdot n\cdot 2\,n}{6\,n^3}\big)+1 = \big(\lim\limits_{n\to+\infty}\frac{2\,n^3}{6\,n^3}\big)+1 = \tfrac{1}{3}+1 = \tfrac{4}{3}$

$\lim\limits_{n\to+\infty} \mathrm{A}_n = \lim\limits_{n\to+\infty}\big(\frac{n\,(n+1)\,(2\,n+1))}{6\,n^3}+1\big) = \big(\lim\limits_{n\to+\infty}\frac{n\,(n+1)\,(2\,n+1))}{6\,n^3}\big)+1$

$= \big(\lim\limits_{n\to+\infty}\frac{n\cdot n\cdot 2\,n}{6\,n^3}\big)+1 = \big(\lim\limits_{n\to+\infty}\frac{2\,n^3}{6\,n^3}\big)+1 = \tfrac{1}{3}+1 = \tfrac{4}{3}$

Pour tout $n \in \mathbb{N}$, on a : $a_n \leqslant \mathcal{A} \leqslant \mathrm{A}_n$.
Par passage à la limite, on obtient : $\lim\limits_{n\to+\infty} a_n \leqslant \mathcal{A} \leqslant \lim\limits_{n\to+\infty} \mathrm{A}_n$
c'est-à-dire $\frac{4}{3} \leqslant \mathcal{A} \leqslant \frac{4}{3}$. On conclut que $\mathcal{A} = \frac{4}{3}$.