11.14 1) $\displaystyle\int_t^1 \frac{1}{\sqrt{x}}\,dx = \int_t^1 x^{-\frac{1}{2}}\,dx = \frac{1}{\frac{1}{2}} x^{\frac{1}{2}}\Big|_t^1 = 2\sqrt{x}\Big|_t^1 = 2\sqrt{1} - 2\sqrt{t} = 2 - 2\sqrt{t}$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\substack{t\to 0\\ t>0}} \int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\substack{t\to 0\\ t>0}} 2 - 2\sqrt{t} = 2 - 2\sqrt{0} = 2 - 0 = 2$$

2) $\displaystyle\int_t^1 \frac{1}{x^2}\,dx = \int_t^1 x^{-2}\,dx = \tfrac{1}{-1}\,x^{-1}\Big|_t^1 = -\frac{1}{x}\Big|_t^1 = -\frac{1}{1} - \left(-\frac{1}{t}\right) = -1 + \frac{1}{t}$

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{\substack{t\to 0\\ t>0}} \frac{1}{x^2}\,dx = \lim_{\substack{t\to 0\\ t>0}} -1 + \frac{1}{t} = -1 + \infty = +\infty$$

Par conséquent, l'intégrale $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$ diverge.

3) $\displaystyle\int_1^t \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_1^t = 2\sqrt{t} - 2\sqrt{1} = 2\sqrt{t} - 2$

$$\int_1^{+\infty} \frac{1}{\sqrt{x}}\,dx = \lim_{t\to+\infty} \int_1^t \frac{1}{\sqrt{x}}\,dx = \lim_{t\to+\infty} 2\sqrt{t} - 2 = 2\sqrt{+\infty} - 2 = +\infty$$

Il en résulte que l'intégrale $\displaystyle\int_1^{+\infty} \frac{1}{\sqrt{x}}\,dx$ diverge.

4) $\displaystyle\int_1^t \frac{1}{\sqrt[3]{x^5}}\,dx = \int_1^t x^{-\frac{5}{3}}\,dx = \frac{1}{-\frac{2}{3}} x^{-\frac{2}{3}}\Big|_1^t = -\frac{3}{2\sqrt[3]{x^2}}\Big|_1^t$

$$= -\frac{3}{2\sqrt[3]{t^2}} - \left(-\frac{3}{2\sqrt[3]{1^2}}\right) = -\frac{3}{2\sqrt[3]{t^2}} + \frac{3}{2}$$

$$\int_1^{+\infty} \frac{1}{\sqrt[3]{x^5}}\,dx = \lim_{t\to+\infty} \int_1^t \frac{1}{\sqrt[3]{x^5}}\,dx = \lim_{t\to+\infty} -\frac{3}{2\sqrt[3]{t^2}} + \frac{3}{2}$$

$$= -\frac{3}{2\sqrt[3]{(+\infty)^2}} + \frac{3}{2} = -0 + \frac{3}{2} = \frac{3}{2}$$

5) $\displaystyle\int_{-1}^2 \frac{1}{x}\,dx = \lim_{\substack{t\to 0\\ t<0}} \int_{-1}^t \frac{1}{x}\,dx + \lim_{\substack{t\to 0\\ t>0}} \int_t^2 \frac{1}{x}\,dx$

$$\lim_{\substack{t\to 0\\ t<0}} \int_{-1}^t \frac{1}{x}\,dx = \lim_{\substack{t\to 0\\ t<0}} \ln\big(|x|\big)\Big|_{-1}^t = \lim_{\substack{t\to 0\\ t<0}} \ln\big(|t|\big) - \ln\big(|-1|\big)$$

$$= \lim_{\substack{t\to 0\\ t<0}} \ln(-t) - \ln(1) = -\infty - 0 = -\infty$$

Il apparaît ainsi que l'intégrale $\displaystyle\int_{-1}^0 \frac{1}{x}\,dx$ diverge, de même que $\displaystyle\int_{-1}^2 \frac{1}{x}\,dx$.

6) Calculons $\displaystyle\int_1^t \frac{\ln(x)}{x^2}\,dx$ par intégration par parties :

$f'(x) = \dfrac{1}{x^2} \qquad f(x) = -\dfrac{1}{x}$

$g(x) = \ln(x) \qquad g'(x) = \dfrac{1}{x}$

$$\int_1^t \frac{\ln(x)}{x^2}\,dx = -\frac{\ln(x)}{x}\Big|_1^t - \int_1^t -\frac{1}{x^2}\,dx = -\frac{\ln(x)}{x}\Big|_1^t - \frac{1}{x}\Big|_1^t = -\frac{\ln(x)+1}{x}\Big|_1^t$$

$$= -\frac{\ln(t)+1}{t} - \left(-\frac{\ln(1)+1}{1}\right) = -\frac{\ln(t)+1}{t} + 1$$

$$\int_1^{+\infty} \frac{\ln(x)}{x^2}\,dx = \lim_{t\to+\infty}\int_1^t \frac{\ln(x)}{x^2}\,dx = \lim_{t\to+\infty} -\frac{\ln(t)+1}{t} + 1$$

$$= -\frac{\ln(+\infty)+1}{+\infty} + 1 = -\frac{+\infty}{+\infty} + 1 : \text{indéterminé}$$

Pour lever cette indétermination, on applique le théorème de Bernouilli-L'Hospital :

$$\lim_{t\to+\infty} -\frac{\ln(t)+1}{t} + 1 = \lim_{t\to+\infty} -\frac{\big(\ln(t)+1\big)'}{(t)'} + 1 = \lim_{t\to+\infty} -\frac{\frac{1}{t}}{1} + 1 = \lim_{t\to+\infty} -\frac{1}{t} + 1 = -0 + 1 = 1$$

7) $$\int_0^{+\infty} \sin(x)\,dx = \lim_{t\to+\infty}\int_0^t \sin(x)\,dx = \lim_{t\to+\infty} -\cos(x)\Big|_0^t$$

$$= \lim_{t\to+\infty} -\cos(t) - \big(-\cos(0)\big) = \lim_{t\to+\infty} -\cos(t) + 1$$

Or cette dernière limite n'existe pas : $\displaystyle\int_0^{+\infty} \sin(x)\,dx$ diverge.

8) $$\int_t^2 \frac{3}{\sqrt{x^3}}\,dx = \int_t^2 3\,x^{-\frac{3}{2}}\,dx = \frac{1}{-\frac{1}{2}}\cdot 3\,x^{-\frac{1}{2}}\Big|_t^2 = -\frac{6}{\sqrt{x}}\Big|_t^2$$

$$= -\frac{6}{\sqrt{2}} - \left(-\frac{6}{\sqrt{t}}\right) = -3\sqrt{2} + \frac{6}{\sqrt{t}}$$

$$\int_0^2 \frac{3}{\sqrt{x^3}}\,dx = \lim_{\substack{t\to0\\t>0}}\int_t^2 \frac{3}{\sqrt{x^3}}\,dx = \lim_{\substack{t\to0\\t>0}} -3\sqrt{2} + \frac{6}{\sqrt{t}} = -3\sqrt{2} + \infty = +\infty$$

Cela signifie que l'intégrale $\displaystyle\int_0^2 \frac{3}{\sqrt{x^3}}\,dx$ diverge.

9) $$\int_0^t x\,e^{-x^2}\,dx = -\tfrac{1}{2}\int_0^t e^{-x^2}\cdot(-2\,x)\,dx = -\tfrac{1}{2}\,e^{-x^2}\Big|_0^t = -\tfrac{1}{2}\,e^{-t^2} - \left(-\tfrac{1}{2}\,e^{-0^2}\right)$$

$$= -\tfrac{1}{2}\,e^{-t^2} + \tfrac{1}{2}$$

$$\int_0^{+\infty} x\,e^{-x^2}\,dx = \lim_{t\to+\infty} \int_0^{+\infty} x\,e^{-x^2}\,dx = \lim_{t\to+\infty} -\tfrac{1}{2}\,e^{-t^2} + \tfrac{1}{2} = -\tfrac{1}{2}\,e^{-\infty} + \tfrac{1}{2}$$

$$= -\tfrac{1}{2}\cdot 0 + \tfrac{1}{2} = \tfrac{1}{2}$$

10) $$\int_0^{+\infty} e^{-x}\,dx = \lim_{t\to+\infty} \int_0^{t} e^{-x}\,dx = \lim_{t\to+\infty} -e^{-x}\Big|_0^t = \lim_{t\to+\infty} -e^{-t} - \big(e^{-0}\big)$$

$$= \lim_{t\to+\infty} -e^{-t} + 1 = -e^{-\infty} + 1 = -0 + 1 = 1$$

11) À l'exercice 3.15 5), on a montré, en intégrant par parties, que $\ln(x)$ admet pour primitive $x\,\big(\ln(x) - 1\big)$.

$$\int_0^{1} \ln(x)\,dx = \lim_{\substack{t\to 0\\ t>0}} \int_t^{1} \ln(x)\,dx = \lim_{\substack{t\to 0\\ t>0}} x\,\big(\ln(x) - 1\big)\Big|_t^1$$

$$= \lim_{\substack{t\to 0\\ t>0}} 1\cdot\big(\ln(1) - 1\big) - t\,\big(\ln(t) - 1\big) = \lim_{\substack{t\to 0\\ t>0}} -1 - 0\cdot\big(\underbrace{-\ln(0)}_{+\infty} - 1\big)$$

Cette dernière limite est indéterminée : on recourt au théorème de Bernouilli-L'Hospital :

$$\lim_{\substack{t\to 0\\ t>0}} t\,\big(\ln(t) - 1\big) = \lim_{\substack{t\to 0\\ t>0}} \frac{\ln(t) - 1}{\frac{1}{t}} = \lim_{\substack{t\to 0\\ t>0}} \frac{\big(\ln(t) - 1\big)'}{\big(\frac{1}{t}\big)'} = \lim_{\substack{t\to 0\\ t>0}} \frac{\frac{1}{t}}{-\frac{1}{t^2}}$$

$$= \lim_{\substack{t\to 0\\ t>0}} -t = 0$$

En définitive $\displaystyle\int_0^{1} \ln(x)\,dx = -1 - \lim_{\substack{t\to 0\\ t>0}} t\,\big(\ln(t) - 1\big) = -1 - 0 = -1$.

12) $$\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to-\infty} \int_t^{0} \frac{1}{x^2+1}\,dx + \lim_{t\to+\infty} \int_0^{t} \frac{1}{x^2+1}\,dx$$

$$= \lim_{t\to-\infty} \arctan x\Big|_t^0 + \lim_{t\to+\infty} \arctan x\Big|_0^t$$

$$= \big(\arctan(0) - \arctan(-\infty)\big) + \big(\arctan(+\infty) - \arctan(0)\big)$$

$$= \big(0 - (-\tfrac{\pi}{2})\big) + \big(\tfrac{\pi}{2} - 0\big) = \tfrac{\pi}{2} + \tfrac{\pi}{2} = \pi$$