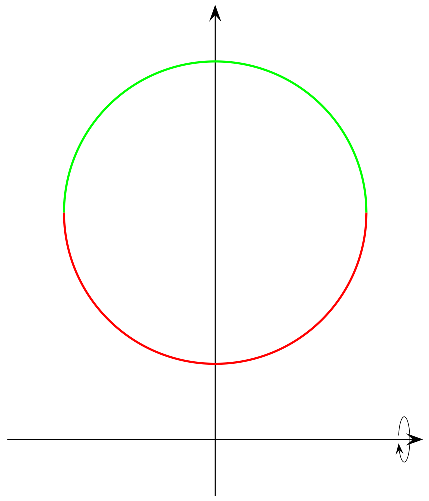
11.25



Posons $f(x) = \mathrm{R} + \sqrt{r^2 - x^2}$ et $g(x) = \mathrm{R} - \sqrt{r^2 - x^2}$.

Alors le volume du tore vaut :

$$\pi \int_{-r}^{r} f^2(x)\,dx - \pi \int_{-r}^{r} g^2(x)\,dx = \pi \left(\int_{-r}^{r} f^2(x)\,dx - \int_{-r}^{r} g^2(x)\,dx \right) =$$

$$\pi \int_{-r}^{r} \big(f^2(x) - g^2(x)\big)\,dx = \pi \int_{-r}^{r} \Big(\big(\mathrm{R} + \sqrt{r^2 - x^2}\big)^2 - \big(\mathrm{R} - \sqrt{r^2 - x^2}\big)^2 \Big)\,dx =$$

$$\pi \int_{-r}^{r} \Big(\big(\mathrm{R}^2 + 2\,\mathrm{R}\,\sqrt{r^2 - x^2} + r^2 - x^2\big) - \big(\mathrm{R}^2 - 2\,\mathrm{R}\,\sqrt{r^2 - x^2} + r^2 - x^2\big) \Big)\,dx =$$

$$\pi \int_{-r}^{r} 4\,\mathrm{R}\,\sqrt{r^2 - x^2}\,dx = 4\,\pi\,\mathrm{R} \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx$$

Effectuons le changement de variable $x = r\,\sin(t)$.

Cette formule donne $\sin(t) = \frac{x}{r}$, puis $t = \arcsin(\frac{x}{r})$.

Les bornes de l'intégrale deviennent donc :

$\arcsin(\frac{-r}{r}) = \arcsin(-1) = -\frac{\pi}{2}$ et $\arcsin(\frac{r}{r}) = \arcsin(1) = \frac{\pi}{2}$.

$$4\,\pi\,\mathrm{R} \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = 4\,\pi\,\mathrm{R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 - \big(r\,\sin(t)\big)^2}\,\big(r\,\sin(t)\big)'\,dt =$$

$$4\,\pi\,\mathrm{R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\sqrt{r^2 - r^2\,\sin^2(t)}}_{r\,\cos(t)} \cdot r\,\cos(t)\,dt = 4\,\pi\,\mathrm{R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2\,\cos^2(t)\,dt =$$

$$4\,\pi\,\mathrm{R}\,r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(t)\,dt = 4\,\pi\,\mathrm{R}\,r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos(2\,t)}{2}\,dt =$$

$$4\,\pi\,\mathrm{R}\,r^2 \cdot \tfrac{1}{2} \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \big(1 + \cos(2\,t)\big)\,dt = 2\,\pi\,\mathrm{R}\,r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \big(1 + \tfrac{1}{2}\,\cos(2\,t) \cdot 2\big)\,dt =$$

$$2\,\pi\,\mathrm{R}\,r^2 \left(t + \tfrac{1}{2}\,\sin(2\,t) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \right) =$$

$2\,\pi\,\mathrm{R}\,r^2\left(\left(\frac{\pi}{2}+\frac{1}{2}\,\sin(2\cdot\frac{\pi}{2})\right)-\left(-\frac{\pi}{2}+\frac{1}{2}\,\sin(2\cdot(-\frac{\pi}{2}))\right)\right)=$

$2\,\pi\,\mathrm{R}\,r^2\left(\left(\frac{\pi}{2}+\frac{1}{2}\,\sin(\pi)\right)-\left(-\frac{\pi}{2}+\sin(-\pi)\right)\right)=2\,\pi\,\mathrm{R}\,r^2\left(\left(\frac{\pi}{2}+0\right)-\left(-\frac{\pi}{2}+0\right)\right)=$

$2\,\pi\,\mathrm{R}\,r^2\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=2\,\pi\,\mathrm{R}\,r^2\,\pi=2\,\pi^2\,\mathrm{R}\,r^2$