10.13 1) $\dfrac{x}{(x-2)^2} = \dfrac{\mathrm{A}}{x-2} + \dfrac{\mathrm{B}}{(x-2)^2} = \dfrac{\mathrm{A}\,(x-2)+\mathrm{B}}{(x-2)^2} = \dfrac{\mathrm{A}\,x+(-2\,\mathrm{A}+\mathrm{B})}{(x-2)^2}$

$$\begin{cases} \mathrm{A} = 1 \\ -2\,\mathrm{A}+\mathrm{B} = 0 \end{cases} \implies \begin{cases} \mathrm{A} = 1 \\ \mathrm{B} = 2 \end{cases}$$

$$\begin{aligned} \int \frac{x}{(x-2)^2}\,dx &= \int \Big(\frac{1}{x-2} + \frac{2}{(x-2)^2}\Big)\,dx \\ &= \int \frac{1}{x-2}\,dx + 2\int (x-2)^{-2}\,dx \\ &= \ln\big(|x-2|\big) + 2\cdot\tfrac{1}{-1}\,(x-2)^{-1} = \ln\big(|x-2|\big) - \frac{2}{x-2} + c \end{aligned}$$

2) $\dfrac{1}{x\,(x-1)\,(x-2)} = \dfrac{\mathrm{A}}{x} + \dfrac{\mathrm{B}}{x-1} + \dfrac{\mathrm{C}}{x-2}$

$$\begin{aligned} &= \frac{\mathrm{A}\,(x-1)\,(x-2) + \mathrm{B}\,x\,(x-2) + \mathrm{C}\,x\,(x-1)}{x\,(x-1)\,(x-2)} \\ &= \frac{(\mathrm{A}+\mathrm{B}+\mathrm{C})\,x^2 - (3\,\mathrm{A}+2\,\mathrm{B}+\mathrm{C})\,x + 2\,\mathrm{A}}{x\,(x-1)\,(x-2)} \end{aligned}$$

$$\begin{cases} \mathrm{A}+\mathrm{B}+\mathrm{C} = 0 \\ 3\,\mathrm{A}+2\,\mathrm{B}+\mathrm{C} = 0 \\ 2\,\mathrm{A} = 1 \end{cases} \implies \begin{cases} \mathrm{B}+\mathrm{C} = -\frac{1}{2} \\ 2\,\mathrm{B}+\mathrm{C} = -\frac{3}{2} \\ \mathrm{A} = \frac{1}{2} \end{cases} \implies$$

$$\begin{cases} \mathrm{B}+\mathrm{C} = -\frac{1}{2} \\ \mathrm{B} = -1 \\ \mathrm{A} = \frac{1}{2} \end{cases} \implies \begin{cases} \mathrm{C} = \frac{1}{2} \\ \mathrm{B} = -1 \\ \mathrm{A} = \frac{1}{2} \end{cases}$$

$$\begin{aligned} \int \frac{1}{x\,(x-1)\,(x-2)}\,dx &= \int \Big(\frac{1}{2\,x} - \frac{1}{x-1} + \frac{1}{2\,(x-2)}\Big)\,dx \\ &= \tfrac{1}{2}\int \frac{1}{x}\,dx + \int \frac{1}{x-1}\,dx + \tfrac{1}{2}\int \frac{1}{x-2}\,dx \\ &= \tfrac{1}{2}\,\ln\big(|x|\big) - \ln\big(|x-1|\big) + \tfrac{1}{2}\,\ln\big(|x-2|\big) + c \end{aligned}$$

3) $\dfrac{2\,x^2+x-2}{x^2\,(x+2)} = \dfrac{\mathrm{A}}{x} + \dfrac{\mathrm{B}}{x^2} + \dfrac{\mathrm{C}}{x+2} = \dfrac{\mathrm{A}\,x\,(x+2) + \mathrm{B}\,(x+2) + \mathrm{C}\,x^2}{x^2\,(x+2)}$

$$= \frac{(\mathrm{A}+\mathrm{C})\,x^2 + (2\,\mathrm{A}+\mathrm{B})\,x + 2\,\mathrm{B}}{x^2\,(x+2)}$$

$$\begin{cases} \mathrm{A}+\mathrm{C} = 2 \\ 2\,\mathrm{A}+\mathrm{B} = 1 \\ 2\,\mathrm{B} = -2 \end{cases} \implies \begin{cases} \mathrm{A}+\mathrm{C} = 2 \\ 2\,\mathrm{A} = 2 \\ \mathrm{B} = -1 \end{cases} \implies \begin{cases} \mathrm{C} = 1 \\ \mathrm{A} = 1 \\ \mathrm{B} = -1 \end{cases}$$

$$\int \frac{2\,x^2+x-2}{x^2\,(x+2)}\,dx = \int \Big(\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x+2}\Big)\,dx$$
$$= \int \frac{1}{x}\,dx - \int \frac{1}{x^2}\,dx + \int \frac{1}{x+2}\,dx$$
$$= \ln\big(|x|\big) + \frac{1}{x} + \ln\big(|x+2|\big) + c$$

4) $$\frac{x+7}{x^2-x-2} = \frac{x+7}{(x-2)\,(x+1)} = \frac{\mathrm{A}}{x-2}+\frac{\mathrm{B}}{x+1} = \frac{\mathrm{A}\,(x+1)+\mathrm{B}\,(x-2)}{(x-2)\,(x+1)}$$
$$= \frac{(\mathrm{A}+\mathrm{B})\,x+(\mathrm{A}-2\,\mathrm{B})}{(x-2)\,(x+1)}$$

$$\begin{cases} \mathrm{A}+\ \mathrm{B}=1 \\ \mathrm{A}-2\,\mathrm{B}=7 \end{cases} \implies \begin{cases} \mathrm{A}+\ \mathrm{B}=\ 1 \\ \quad 3\,\mathrm{B}=-6 \end{cases} \implies \begin{cases} \mathrm{A}=\ 3 \\ \mathrm{B}=-2 \end{cases}$$

$$\int \frac{x+7}{x^2-x-2}\,dx = \int \left(\frac{3}{x-2}-\frac{2}{x+1}\right)\,dx$$
$$= 3\int \frac{1}{x-2}\,dx - 2\int \frac{1}{x+1}\,dx$$
$$= 3\,\ln\big(|x-2|\big) - 2\,\ln\big(|x+1|\big) + c$$

5)
$$\begin{array}{r|l} x^2-4\,x & x^2-4\,x+3 \\ \hline -x^2+4\,x-3 & 1 \\ \hline -3 & \end{array}$$

$$\frac{x^2-4\,x}{x^2-4\,x+3} = 1-\frac{3}{x^2-4\,x+3} = 1-\frac{3}{(x-1)\,(x-3)}$$
$$= 1+\frac{\mathrm{A}}{x-1}+\frac{\mathrm{B}}{x-3} = 1+\frac{\mathrm{A}\,(x-3)+\mathrm{B}\,(x-1)}{(x-1)\,(x-3)}$$
$$= 1+\frac{(\mathrm{A}+\mathrm{B})\,x-(3\,\mathrm{A}+\mathrm{B})}{(x-1)\,(x-3)}$$

$$\begin{cases} \mathrm{A}+\mathrm{B}=0 \\ 3\,\mathrm{A}+\mathrm{B}=3 \end{cases} \implies \begin{cases} \mathrm{A}+\mathrm{B}=0 \\ 2\,\mathrm{A}\quad\ =3 \end{cases} \implies \begin{cases} \mathrm{A}=\ \frac{3}{2} \\ \mathrm{B}=-\frac{3}{2} \end{cases}$$

$$\int \frac{x^2-4\,x}{x^2-4\,x+3}\,dx = \int \left(1+\frac{3}{2\,(x-1}-\frac{3}{2\,(x-3)}\right)\,dx$$
$$= \int 1\,dx + \tfrac{3}{2}\int \frac{1}{x-1}\,dx - \tfrac{3}{2}\int \frac{1}{x-3}\,dx$$
$$= x + \tfrac{3}{2}\,\ln\big(|x-1|\big) - \tfrac{3}{2}\,\ln\big(|x-3|\big) + c$$

6) $$\frac{8\,x-8}{x^3-4\,x}=\frac{8\,x-8}{x\,(x-2)\,(x+2)}=\frac{\mathrm{A}}{x}+\frac{\mathrm{B}}{x-2}+\frac{\mathrm{C}}{x+2}$$
$$=\frac{\mathrm{A}\,(x-2)\,(x+2)+\mathrm{B}\,x\,(x+2)+\mathrm{C}\,x\,(x-2)}{x\,(x-2)\,(x+2)}$$
$$=\frac{(\mathrm{A}+\mathrm{B}+\mathrm{C})\,x^2+(2\,\mathrm{B}-2\,\mathrm{C})\,x+(-4\,\mathrm{A})}{x\,(x-2)\,(x+2)}$$

$$\begin{cases} \mathrm{A}+\mathrm{B}+\mathrm{C}=0 \\ 2\,\mathrm{B}-2\,\mathrm{C}=8 \\ -4\,\mathrm{A}=-8 \end{cases}\Longrightarrow\begin{cases} \mathrm{B}+\mathrm{C}=-2 \\ \mathrm{B}-\mathrm{C}=4 \\ \mathrm{A}=2 \end{cases}$$

$$\Longrightarrow\begin{cases} \mathrm{B}+\mathrm{C}=-2 \\ 2\,\mathrm{B}=2 \\ \mathrm{A}=2 \end{cases}\Longrightarrow\begin{cases} \mathrm{C}=-3 \\ \mathrm{B}=1 \\ \mathrm{A}=2 \end{cases}$$

$$\int\frac{8\,x-8}{x^3-4\,x}\,dx=\int\left(\frac{2}{x}+\frac{1}{x-2}-\frac{3}{x+2}\right)dx$$
$$=2\int\frac{1}{x}\,dx+\int\frac{1}{x-2}\,dx-3\int\frac{1}{x+2}\,dx$$
$$=2\,\ln\bigl(|x|\bigr)+\ln\bigl(|x-2|\bigr)-3\,\ln\bigl(|x+2|\bigr)+c$$

7) $$\frac{2-3\,x}{x^2-3\,x+2}=\frac{2-3\,x}{(x-1)\,(x-2)}=\frac{\mathrm{A}}{x-1}+\frac{\mathrm{B}}{x-2}=\frac{\mathrm{A}\,(x-2)+\mathrm{B}\,(x-1)}{(x-1)\,(x-2)}$$
$$=\frac{(\mathrm{A}+\mathrm{B})\,x+(-2\,\mathrm{A}-\mathrm{B})}{(x-1)\,(x-2)}$$

$$\begin{cases} \mathrm{A}+\mathrm{B}=-3 \\ -2\,\mathrm{A}-\mathrm{B}=2 \end{cases}\Longrightarrow\begin{cases} \mathrm{A}+\mathrm{B}=-3 \\ -\mathrm{A}=-1 \end{cases}\Longrightarrow\begin{cases} \mathrm{B}=-4 \\ \mathrm{A}=1 \end{cases}$$

$$\int\frac{2-3\,x}{x^2-3\,x+2}\,dx=\int\left(\frac{1}{x-1}-\frac{4}{x-2}\right)dx$$
$$=\int\frac{1}{x-1}\,dx-4\int\frac{1}{x-2}\,dx$$
$$=\ln\bigl(|x-1|\bigr)-4\,\ln\bigl(|x-2|\bigr)+c$$

8) $$\frac{x^2-4\,x-1}{x^3-x}=\frac{x^2-4\,x-1}{x\,(x-1)\,(x+1)}=\frac{\mathrm{A}}{x}+\frac{\mathrm{B}}{x-1}+\frac{\mathrm{C}}{x+1}$$
$$=\frac{\mathrm{A}\,(x-1)\,(x+1)+\mathrm{B}\,x\,(x+1)+\mathrm{C}\,x\,(x-1)}{x\,(x-1)\,(x+1)}$$
$$=\frac{(\mathrm{A}+\mathrm{B}+\mathrm{C})\,x^2+(\mathrm{B}-\mathrm{C})\,x-\mathrm{A}}{x\,(x-1)\,(x+1)}$$

$$\begin{cases} \mathrm{A}+\mathrm{B}+\mathrm{C}=1 \\ \mathrm{B}-\mathrm{C}=-4 \\ -\mathrm{A}=-1 \end{cases}\Longrightarrow\begin{cases} \mathrm{B}+\mathrm{C}=0 \\ \mathrm{B}-\mathrm{C}=-4 \\ \mathrm{A}=1 \end{cases}\Longrightarrow\begin{cases} \mathrm{C}=2 \\ \mathrm{B}=-2 \\ \mathrm{A}=1 \end{cases}$$

$$\int \frac{x^2-4\,x-1}{x^3-x}\,dx = \int \left(\frac{1}{x}-\frac{2}{x-1}+\frac{2}{x+1}\right)\,dx$$
$$= \int \frac{1}{x}\,dx - 2\int \frac{1}{x-1}\,dx + 2\int \frac{1}{x+1}\,dx$$
$$= \ln\big(|x|\big) - 2\,\ln\big(|x-1|\big) + 2\,\ln\big(|x+1|\big) + c$$

9) $$\frac{3\,x+1}{x\,(x-1)^3} = \frac{\mathrm{A}}{x} + \frac{\mathrm{B}}{x-1} + \frac{\mathrm{C}}{(x-1)^2} + \frac{\mathrm{D}}{(x-1)^3}$$
$$= \frac{\mathrm{A}\,(x-1)^3 + \mathrm{B}\,x\,(x-1)^2 + \mathrm{C}\,x\,(x-1) + \mathrm{D}\,x}{x\,(x-1)^3}$$
$$= \frac{(\mathrm{A}+\mathrm{B})\,x^3 + (-3\,\mathrm{A}-2\,\mathrm{B}+\mathrm{C})\,x^2 + (3\,\mathrm{A}+\mathrm{B}-\mathrm{C}+\mathrm{D})\,x - \mathrm{A}}{x\,(x-1)^3}$$

$$\begin{cases} \mathrm{A}+\mathrm{B} = 0 \\ -3\,\mathrm{A}-2\,\mathrm{B}+\mathrm{C} = 0 \\ 3\,\mathrm{A}+\mathrm{B}-\mathrm{C}+\mathrm{D} = 3 \\ -\mathrm{A} = 1 \end{cases} \Longrightarrow \begin{cases} \mathrm{B} = 1 \\ \mathrm{C} = -1 \\ \mathrm{D} = 4 \\ \mathrm{A} = -1 \end{cases}$$

$$\int \frac{3\,x+1}{x\,(x-1)^3}\,dx = \int \left(-\frac{1}{x}+\frac{1}{x-1}-\frac{1}{(x-1)^2}+\frac{4}{(x-1)^3}\right)\,dx$$
$$= -\int \frac{1}{x}\,dx + \int \frac{1}{x-1}\,dx - \int \frac{1}{(x-1)^2}\,dx + 4\int \frac{1}{(x-1)^3}\,dx$$
$$= -\ln\big(|x|\big) + \ln\big(|x-1|\big) - \frac{1}{-1}\cdot\frac{1}{x-1} + 4\cdot\frac{1}{-2}\cdot\frac{1}{(x-1)^2}$$
$$= -\ln\big(|x|\big) + \ln\big(|x-1|\big) + \frac{1}{x-1} - \frac{2}{(x-1)^2} + c$$

10) $$\frac{4\,x}{x^4-1} = \frac{4\,x}{(x^2-1)\,(x^2+1)} = \frac{4\,x}{(x-1)\,(x+1)\,(x^2+1)}$$
$$= \frac{\mathrm{A}}{x-1} + \frac{\mathrm{B}}{x+1} + \frac{\mathrm{C}\,x+\mathrm{D}}{x^2+1}$$
$$= \frac{\mathrm{A}\,(x+1)\,(x^2+1) + \mathrm{B}\,(x-1)\,(x^2+1) + (\mathrm{C}\,x+\mathrm{D})\,(x-1)\,(x+1)}{x\,(x-1)\,(x+1)\,(x^2+1)}$$
$$= \frac{(\mathrm{A}+\mathrm{B}+\mathrm{C})\,x^3 + (\mathrm{A}-\mathrm{B}+\mathrm{D})\,x^2 + (\mathrm{A}+\mathrm{B}-\mathrm{C})\,x + (\mathrm{A}-\mathrm{B}-\mathrm{D})}{x\,(x-1)\,(x+1)\,(x^2+1)}$$

$$\begin{cases} \mathrm{A}+\mathrm{B}+\mathrm{C} = 0 \\ \mathrm{A}-\mathrm{B}+\mathrm{D} = 0 \\ \mathrm{A}+\mathrm{B}-\mathrm{C} = 4 \\ \mathrm{A}-\mathrm{B}-\mathrm{D} = 0 \end{cases} \Longrightarrow \begin{cases} \mathrm{A}+\mathrm{B}+\mathrm{C} = 0 \\ -2\,\mathrm{B}-\mathrm{C}+\mathrm{D} = 0 \\ -2\,\mathrm{C} = 4 \\ -2\,\mathrm{B}-\mathrm{C}-\mathrm{D} = 0 \end{cases} \Longrightarrow$$

$$\begin{cases} \mathrm{C} = -2 \\ \mathrm{A}+\mathrm{B} = 2 \\ -2\,\mathrm{B}+\mathrm{D} = -2 \\ -2\,\mathrm{B}-\mathrm{D} = -2 \end{cases} \Longrightarrow \begin{cases} \mathrm{C} = -2 \\ \mathrm{A}+\mathrm{B} = 2 \\ -2\,\mathrm{B}+\mathrm{D} = -2 \\ -4\,\mathrm{B} = -4 \end{cases} \Longrightarrow \begin{cases} \mathrm{C} = -2 \\ \mathrm{A} = 1 \\ \mathrm{D} = 0 \\ \mathrm{B} = 1 \end{cases}$$

$$\begin{aligned}\int \frac{4\,x}{x^4-1}\,dx &= \int \left(\frac{1}{x-1}+\frac{1}{x+1}-\frac{2\,x}{x^2+1}\right)\,dx\\ &= \int \frac{1}{x-1}\,dx+\frac{1}{x+1}\,dx-\int \frac{1}{x^2+1}\cdot \underbrace{2\,x}_{(x^2+1)'}\,dx\\ &= \ln\bigl(|x-1|\bigr)+\ln\bigl(|x+1|\bigr)-\ln(x^2+1)+c\end{aligned}$$