10.15 1) $f'(x) = \sin(2\,x)$

$$f(x) = \int \sin(2\,x)\,dx = \int \sin(2\,x)\cdot 2\cdot \tfrac{1}{2}\,dx = \tfrac{1}{2}\int \sin(2\,x)\cdot 2\,dx$$
$$= \tfrac{1}{2}\big(-\cos(2\,x)\big) = -\tfrac{1}{2}\,\cos(2\,x)$$

$g(x) = x$

$g'(x) = 1$

$$\int x\,\sin(2\,x)\,dx = -\tfrac{1}{2}\,\cos(2\,x)\cdot x - \int -\tfrac{1}{2}\,\cos(2\,x)\cdot 1\,dx$$
$$= -\tfrac{1}{2}\,x\,\cos(2\,x) + \tfrac{1}{2}\int \cos(2\,x)\,dx$$
$$= -\frac{1}{2}\,x\,\cos(2\,x) + \frac{1}{2}\int \cos(2\,x)\cdot 2\cdot \tfrac{1}{2}\,dx$$
$$= -\tfrac{1}{2}\,x\,\cos(2\,x) + \tfrac{1}{2}\cdot\tfrac{1}{2}\int \cos(2\,x)\cdot 2\,dx$$
$$= -\tfrac{1}{2}\,x\,\cos(2\,x) + \tfrac{1}{4}\,\sin(2\,x) + c$$

2) $f'(x) = e^x \qquad f(x) = e^x$

$g(x) = x \qquad g'(x) = 1$

$$\int x\,e^x\,dx = x\,e^x - \int e^x\,dx = x\,e^x - e^x = (x-1)\,e^x + c$$

3) $f'(x) = e^{3x}$

$$f(x) = \int e^{3x}\,dx = \int e^{3x}\cdot 3\cdot\tfrac{1}{3}\,dx = \tfrac{1}{3}\int e^{3x}\cdot 3\,dx = \tfrac{1}{3}\,e^{3x}$$

$g(x) = 3\,x^2 \qquad g'(x) = 6\,x$

$$\int 3\,x^2\,e^{3x}\,dx = \tfrac{1}{3}\,e^{3x}\cdot 3\,x^2 - \int \tfrac{1}{3}\,e^{3x}\cdot 6\,x\,dx = x^2\,e^{3x} - \int 2\,x\,e^{3x}\,dx$$

Pour calculer $\int 2\,x\,e^{3x}\,dx$, on fait à nouveau une intégration par parties :

$f'(x) = e^{3x} \qquad f(x) = \tfrac{1}{3}\,e^{3x}$

$g(x) = 2\,x \qquad g'(x) = 2$

$$\int 2\,x\,e^{3x}\,dx = \tfrac{1}{3}\,e^{3x}\cdot 2\,x - \int \tfrac{1}{3}\,e^{3x}\cdot 2\,dx = \tfrac{2}{3}\,x\,e^{3x} - \tfrac{2}{3}\int e^{3x}\,dx$$
$$= \tfrac{2}{3}\,x\,e^{3x} - \tfrac{2}{3}\cdot\tfrac{1}{3}\,e^{3x} = \tfrac{2}{3}\,x\,e^{3x} - \tfrac{2}{9}\,e^{3x}$$

Finalement, $\displaystyle\int 3\,x^2\,e^{3x}\,dx = x^2\,e^{3x} - \left(\tfrac{2}{3}\,x\,e^{3x} - \tfrac{2}{9}\,e^{3x}\right)$

$$= x^2\,e^{3x} - \tfrac{2}{3}\,x\,e^{3x} + \tfrac{2}{9}\,e^{3x}$$
$$= \tfrac{1}{9}\,(9\,x^2 - 6\,x + 2)\,e^{3x} + c$$

4) $f'(x) = \cos(x) \qquad f(x) = \sin(x)$

$g(x) = x^2 + 1 \qquad g'(x) = 2\,x$

$$\int (x^2+1)\,\cos(x)\,dx = (x^2+1)\,\sin(x) - \int 2\,x\,\sin(x)\,dx$$

Pour calculer $\displaystyle\int 2\,x\,\sin(x)\,dx$, on procède à une nouvelle intégration par parties :

$f'(x) = \sin(x) \qquad f(x) = -\cos(x)$

$g(x) = 2\,x \qquad g'(x) = 2$

$$\int 2\,x\,\sin(x)\,dx = -2\,x\,\cos(x) - \int -2\,\cos(x)\,dx = -2\,x\,\cos(x) + 2\,\sin(x)$$

Donc $\displaystyle\int (x^2+1)\,\cos(x)\,dx = (x^2+1)\,\sin(x) - \big(-2\,x\,\cos(x) + 2\,\sin(x)\big)$

$$= (x^2-1)\,\sin(x) + 2\,x\,\cos(x) + c$$

5) $f'(x) = 1 \qquad f(x) = x$

$g(x) = \ln(x) \qquad g'(x) = \dfrac{1}{x}$

$$\int \ln(x)\,dx = x\,\ln(x) - \int x \cdot \frac{1}{x}\,dx = x\,\ln(x) - \int 1\,dx = x\,\ln(x) - x$$

$$= x\,\big(\ln(x) - 1\big) + c$$

6) $f'(x) = \sqrt{x+1}$

$$f(x) = \int \sqrt{x+1}\,dx = \int (x+1)^{\frac{1}{2}}\,dx = \frac{1}{\frac{3}{2}}\,(x+1)^{\frac{3}{2}} = \tfrac{2}{3}\,\sqrt{(x+1)^3}$$

$$= \tfrac{2}{3}\,(x+1)\,\sqrt{x+1}$$

$g(x) = x \qquad g'(x) = 1$

$$\int x\,\sqrt{x+1}\,dx = \tfrac{2}{3}\,x\,(x+1)\,\sqrt{x+1} - \int \tfrac{2}{3}\,(x+1)\,\sqrt{x+1}\,dx$$

$$= \tfrac{2}{3}\,x\,(x+1)\,\sqrt{x+1} - \tfrac{2}{3}\int (x+1)^{\frac{3}{2}}\,dx$$

$$= \tfrac{2}{3}\,x\,(x+1)\,\sqrt{x+1} - \tfrac{2}{3} \cdot \frac{1}{\frac{5}{2}}\,(x+1)^{\frac{5}{2}}$$

$$= \tfrac{2}{3}\,x\,(x+1)\,\sqrt{x+1} - \tfrac{4}{15}\,(x+1)^2\,\sqrt{x+1}$$

$$= \tfrac{2}{15}\,(x+1)\,\sqrt{x+1}\,\big(5\,x - 2\,(x+1)\big)$$

$$= \tfrac{2}{15}\,(3\,x-2)\,(x+1)\,\sqrt{x+1} + c$$

7) $f'(x) = 1 \qquad f(x) = x$

$g(x) = \arcsin(x) \qquad g'(x) = \dfrac{1}{\sqrt{1-x^2}}$

$$\int \arcsin(x)\,dx = x\,\arcsin(x) - \int \frac{1}{\sqrt{1-x^2}} \cdot x\,dx$$
$$= x\,\arcsin(x) - (-\tfrac{1}{2}) \int (1-x^2)^{-\frac{1}{2}} \cdot (-2\,x)\,dx$$
$$= x\,\arcsin(x) + \tfrac{1}{2} \cdot \frac{1}{\frac{1}{2}}\,(1-x^2)^{\frac{1}{2}}$$
$$= x\,\arcsin(x) + \sqrt{1-x^2} + c$$

8) $f'(x) = \cos(x) \qquad f(x) = \sin(x)$

$g(x) = e^x \qquad g'(x) = e^x$

$$\int e^x\,\cos(x)\,dx = e^x\,\sin(x) - \int e^x\,\sin(x)\,dx$$

Pour calculer $\int e^x\,\sin(x)\,dx$, on recourt derechef à une intégration par parties :

$f'(x) = \sin(x) \qquad f(x) = -\cos(x)$

$g(x) = e^x \qquad g'(x) = e^x$

$$\int e^x\,\sin(x)\,dx = -e^x\,\cos(x) - \int -e^x\,\cos(x)\,dx = -e^x\,\cos(x) + \int e^x\,\cos(x)\,dx$$

Ainsi $\displaystyle\int e^x\,\cos(x)\,dx = e^x\,\sin(x) - \Big(-e^x\,\cos(x) + \int e^x\,\cos(x)\,dx\Big)$

$$= e^x\,\big(\sin(x) + \cos(x)\big) - \int e^x\,\cos(x)\,dx$$

Il en résulte $2 \displaystyle\int e^x\,\cos(x)\,dx = e^x\,\big(\sin(x) + \cos(x)\big)$

d'où finalement $\displaystyle\int e^x\,\cos(x)\,dx = \tfrac{1}{2}\,e^x\,\big(\sin(x) + \cos(x)\big) + c$

9) $f'(x) = x \qquad f(x) = \tfrac{1}{2}\,x^2$

$g(x) = \ln(x) \qquad g'(x) = \dfrac{1}{x}$

$$\int x\,\ln(x)\,dx = \tfrac{1}{2}\,x^2\,\ln(x) - \int \tfrac{1}{2}\,x^2 \cdot \frac{1}{x}\,dx = \tfrac{1}{2}\,x^2\,\ln(x) - \tfrac{1}{2} \int x\,dx$$
$$= \tfrac{1}{2}\,x^2\,\ln(x) - \tfrac{1}{4}\,x^2 = \tfrac{1}{4}\,x^2\,\big(2\,\ln(x) - 1\big) + c$$

10) $f'(x) = e^{-x}$ $\qquad f(x) = -e^{-x}$

$g(x) = x^2$ $\qquad g'(x) = 2\,x$

$$\int x^2\,e^{-x}\,dx = -x^2\,e^{-x} - \int -2\,x\,e^{-x}\,dx = -x^2\,e^{-x} + 2\int x\,e^{-x}\,dx$$

Pour calculer $\displaystyle\int x\,e^{-x}\,dx$, on procède encore par intégration par parties :

$f'(x) = e^{-x}$ $\qquad f(x) = -e^{-x}$

$g(x) = x$ $\qquad g'(x) = 1$

$$\int x\,e^{-x}\,dx = -x\,e^{-x} - \int -e^{-x}\,dx = -x\,e^{-x} - e^{-x}$$

D'où $\displaystyle\int x^2\,e^{-x}\,dx = -x^2\,e^{-x} + 2\left(-x\,e^{-x} - e^{-x}\right) = -(x^2 + 2\,x + 2)\,e^{-x} + c$