10.16 1) $$\int x\sqrt{x+2}\,dx = \int (t^2-2)\sqrt{(t^2-2)+2}\cdot(t^2-2)'\,dt$$
$$= \int (t^2-2)\sqrt{t^2}\cdot 2\,t\,dt = \int (t^2-2)\cdot 2\,t^2\,dt$$
$$= \int (2\,t^4-4\,t^2)\,dt = \tfrac{2}{5}\,t^5-\tfrac{4}{3}\,t^3$$

La formule de changement de variable $x=t^2-2$ implique $t^2=x+2$, puis $t=\sqrt{x+2}$. Il en résulte :

$$\int x\sqrt{x+2}\,dx = \tfrac{2}{5}\,(\sqrt{x+2})^5-\tfrac{4}{3}\,(\sqrt{x+2})^3$$
$$= \tfrac{2}{5}\,(x+2)^2\sqrt{x+2}-\tfrac{4}{3}\,(x+2)\sqrt{x+2}$$
$$= \tfrac{2}{15}\,(x+2)\sqrt{x+2}\,\big(3\,(x+2)-10\big)$$
$$= \tfrac{2}{15}\,(x+2)\sqrt{x+2}\,(3\,x-4)+c$$

2) $$\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = \int \frac{\sqrt{(t-1)^2}}{1+\sqrt{(t-1)^2}}\cdot\big((t-1)^2\big)'\,dt$$
$$= \int \frac{t-1}{1+t-1}\cdot 2\,(t-1)\cdot 1\,dt = 2\int \frac{(t-1)^2}{t}\,dt$$
$$= 2\int\Big(\frac{t^2}{t}-\frac{2\,t}{t}+\frac{1}{t}\Big)\,dt = 2\int\Big(t-2+\frac{1}{t}\Big)\,dt$$
$$= 2\,\Big(\tfrac{1}{2}\,t^2-2\,t+\ln\big(|t|\big)\Big) = t^2-4\,t+2\,\ln\big(|t|\big)$$

L'égalité $x=(t-1)^2$ donne $\sqrt{x}=t-1$, puis $1+\sqrt{x}=t$. Il en découle :

$$\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = (1+\sqrt{x})^2-4\,(1+\sqrt{x})+2\,\ln\big(|1+\sqrt{x}|\big)$$
$$= 1+2\sqrt{x}+x-4-4\sqrt{x}+2\,\ln(1+\sqrt{x})$$
$$= x-2\sqrt{x}-3+2\,\ln(1+\sqrt{x})+c$$

3) $$\int \frac{2\,x+1}{\sqrt{x+1}}\,dx = \int \frac{2\,(t^2-1)+1}{\sqrt{(t^2-1)+1}}\cdot(t^2-1)'\,dt = \int \frac{2\,t^2-1}{\sqrt{t^2}}\cdot 2\,t\,dt$$
$$= \int \frac{4\,t^3-2\,t}{t}\,dt = \int (4\,t^2-2)\,dt = \tfrac{4}{3}\,t^3-2\,t$$

La formule $x=t^2-1$ implique $x+1=t^2$, puis $\sqrt{x+1}=t$. Donc :

$$\int \frac{2\,x+1}{\sqrt{x+1}}\,dx = \tfrac{4}{3}\,(\sqrt{x+1})^3-2\sqrt{x+1} = \tfrac{4}{3}\,(x+1)\sqrt{x+1}-2\sqrt{x+1}$$
$$= \tfrac{2}{3}\sqrt{x+1}\,\big(2\,(x+1)-3\big) = \tfrac{2}{3}\sqrt{x+1}\,(2\,x-1)+c$$

4) $$\int \frac{\arctan^2(x)}{1+x^2}\,dx = \int \frac{\arctan^2\big(\tan(t)\big)}{1+\tan^2(t)} \cdot \big(\tan(t)\big)'\,dt$$
$$= \int \frac{t^2}{1+\tan^2(t)} \cdot \big(1+\tan^2(t)\big)\,dt = \int t^2\,dt = \tfrac{1}{3}\,t^3$$

La formule du changement de variables $x = \tan(t)$ donne $\arctan(x) = t$.
Par conséquent, $\displaystyle\int \frac{\arctan^2(x)}{1+x^2}\,dx = \tfrac{1}{3}\,\arctan^3(x) + c$

5) $$\int \frac{1}{2\,x^2 - 2\,x + 1}\,dx = \int \frac{1}{2\left(\frac{1}{2}\,(t+1)\right)^2 - 2 \cdot \frac{1}{2}\,(t+1) + 1} \cdot \big(\tfrac{1}{2}\,(t+1)\big)'\,dt$$
$$= \int \frac{1}{\frac{1}{2}\,t^2 + t + \frac{1}{2} - t - 1 + 1} \cdot \tfrac{1}{2}\,dt$$
$$= \int \frac{1}{\frac{1}{2}\,(t^2+1)} \cdot \tfrac{1}{2}\,dt = \int \frac{1}{t^2+1}\,dt = \arctan(t)$$

À partir de $x = \frac{1}{2}\,(t+1)$, on tire que $2\,x = t+1$ et que $2\,x - 1 = t$. Ainsi :
$\displaystyle\int \frac{1}{2\,x^2 - 2\,x + 1}\,dx = \arctan(2\,x - 1) + c$

6) $$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2(t)} \cdot \big(\sin(t)\big)'\,dt = \int \cos(t) \cdot \cos(t)\,dt$$
$$= \int \cos^2(t)\,dt = \int \frac{1+\cos(2\,t)}{2}\,dt = \tfrac{1}{2} \int \big(1+\cos(2\,t)\big)\,dt$$
$$= \tfrac{1}{2} \int 1\,dt + \tfrac{1}{2} \int \cos(2\,t)\,dt$$
$$= \tfrac{1}{2} \int 1\,dt + \tfrac{1}{4} \int \cos(2\,t) \cdot 2\,dt = \tfrac{1}{2}\,t + \tfrac{1}{4}\,\sin(2\,t)$$
$$= \tfrac{1}{2}\,t + \tfrac{1}{4} \cdot 2\,\sin(t)\,\cos(t) = \tfrac{1}{2}\,t + \tfrac{1}{2}\,\sin(t)\,\cos(t)$$
$$= \tfrac{1}{2}\,t + \tfrac{1}{2}\,\sin(t)\,\sqrt{1-\sin^2(t)}$$

La formule $x = \sin(t)$ délivre $\arcsin(x) = t$. On en conclut que :
$\displaystyle\int \sqrt{1-x^2}\,dx = \tfrac{1}{2}\,\arcsin(x) + \tfrac{1}{2}\,x\,\sqrt{1-x^2} + c$