5.10 1) La relation de récurrence $u_n = 2\,(u_{n-1} - u_{n-2})$ donne :

$\lambda^2 = 2\,(\lambda - 1)$

$\lambda^2 - 2\,\lambda + 2 = 0$

$\Delta = (-2)^2 - 4 \cdot 1 \cdot 2 = -4 = 4\,i^2$

$\lambda = \frac{-(-2) \pm 2\,i}{2 \cdot 1} = 1 \pm i$

2) On en tire que $u_n = a\,(1+i)^n + b\,(1-i)^n$.

Déterminons les coefficients a et b à l'aide des conditions initiales :

$$\begin{cases} 1 = u_0 = a\,(1+i)^0 + b\,(1-i)^0 = a + b \\ 2 = u_1 = a\,(1+i)^1 + b\,(1-i)^1 = (1+i)\,a + (1-i)\,b \end{cases}$$

Résolvons ce système :

$$\begin{cases} a + b = 1 \\ (1+i)\,a + (1-i)\,b = 2 \end{cases} \xRightarrow{\mathrm{L}_2 \to \mathrm{L}_2 - (1+i)\,\mathrm{L}_1} \begin{cases} a + b = 1 \\ \quad -2\,i\,b = 1 - i \end{cases}$$

$$\xRightarrow{\mathrm{L}_1 \to 2\,i\,\mathrm{L}_1} \begin{cases} 2\,i\,a = 1 + i \\ \quad -2\,i\,b = 1 - i \end{cases} \xRightarrow[\mathrm{L}_2 \to -1/2\,i\,\mathrm{L}_2]{\mathrm{L}_1 \to 1/2\,i\,\mathrm{L}_1} \begin{cases} a = \frac{1+i}{2\,i} = \frac{(1+i)\,(-i)}{2\,i\,(-i)} = \frac{1-i}{2} \\ b = \frac{1-i}{-2\,i} = \frac{(1-i)\,i}{-2\,i \cdot i} = \frac{1+i}{2} \end{cases}$$

On a obtenu $u_n = \frac{1}{2}\,(1-i)\,(1+i)^n + \frac{1}{2}\,(1+i)\,(1-i)^n$.

3) (a) $|1+i| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i\right) = \sqrt{2}\left(\cos(\frac{\pi}{4}) + i\,\sin(\frac{\pi}{4})\right)$

$|1-i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,i\right) = \sqrt{2}\left(\cos(-\frac{\pi}{4}) + i\,\sin(-\frac{\pi}{4})\right)$

(b) Rappelons la formule de Moivre :

$$\Big(r\,\big(\cos(\varphi) + i\,\sin(\varphi)\big)\Big)^n = r^n\,\big(\cos(n\,\varphi) + i\,\sin(n\,\varphi)\big)$$

Utilisons cette formule pour calculer u_n :

$$\begin{aligned} u_n &= \tfrac{1}{2}\,(1-i)\,(1+i)^n + \tfrac{1}{2}\,(1+i)\,(1-i)^n \\ &= \tfrac{1}{2}\,(1-i)\,\Big(\sqrt{2}\,(\cos(\tfrac{\pi}{4}) + i\,\sin(\tfrac{\pi}{4}))\Big)^n + \tfrac{1}{2}\,(1+i)\,\Big(\sqrt{2}\,(\cos(-\tfrac{\pi}{4}) + i\,\sin(-\tfrac{\pi}{4}))\Big)^n \\ &= \tfrac{1}{2}\,(1-i)\,(\sqrt{2})^n\,(\cos(\tfrac{n\,\pi}{4}) + i\,\sin(\tfrac{n\,\pi}{4})) + \tfrac{1}{2}\,(1+i)\,(\sqrt{2})^n\,(\cos(-\tfrac{n\,\pi}{4}) + i\,\sin(-\tfrac{n\,\pi}{4})) \end{aligned}$$

En utilisant que $\cos(-\alpha) = \cos(\alpha)$ et $\sin(-\alpha) = -\sin(\alpha)$, on obtient :

$$\begin{aligned} u_n &= \tfrac{1}{2}\,(1-i)\,(\sqrt{2})^n\,(\cos(\tfrac{n\,\pi}{4}) + i\,\sin(\tfrac{n\,\pi}{4})) + \tfrac{1}{2}\,(1+i)\,(\sqrt{2})^n\,(\cos(\tfrac{n\,\pi}{4}) - i\,\sin(\tfrac{n\,\pi}{4})) \\ &= \tfrac{1}{2}\,(\sqrt{2})^n\Big((1-i)\,(\cos(\tfrac{n\,\pi}{4}) + i\,\sin(\tfrac{n\,\pi}{4})) + (1+i)\,(\cos(\tfrac{n\,\pi}{4}) - i\,\sin(\tfrac{n\,\pi}{4}))\Big) \\ &= \tfrac{1}{2}\,(\sqrt{2})^n\,(\cos(\tfrac{n\,\pi}{4}) + i\,\sin(\tfrac{n\,\pi}{4}) - i\,\cos(\tfrac{n\,\pi}{4}) + \sin(\tfrac{n\,\pi}{4}) + \cos(\tfrac{n\,\pi}{4}) - i\,\sin(\tfrac{n\,\pi}{4}) \\ &\qquad + i\,\cos(\tfrac{n\,\pi}{4}) + \sin(\tfrac{n\,\pi}{4})) \\ &= \tfrac{1}{2}\,(\sqrt{2})^n\,(2\,\cos(\tfrac{n\,\pi}{4}) + 2\,\sin(\tfrac{n\,\pi}{4})) \\ &= (\sqrt{2})^n\,(\cos(\tfrac{n\,\pi}{4}) + \sin(\tfrac{n\,\pi}{4})) \end{aligned}$$