5.14

1) $$2\,(\sqrt{k+1}-\sqrt{k}) = 2\cdot\frac{(\sqrt{k+1}-\sqrt{k})\,(\sqrt{k+1}+\sqrt{k})}{\sqrt{k+1}+\sqrt{k}} = 2\cdot\frac{(k+1)-k}{\sqrt{k+1}+\sqrt{k}}$$
$$= 2\cdot\frac{1}{\sqrt{k+1}+\sqrt{k}} < 2\cdot\frac{1}{\sqrt{k}+\sqrt{k}} = 2\cdot\frac{1}{2\sqrt{k}} = \frac{1}{\sqrt{k}}$$

2) $$\sum_{k=1}^{n}\frac{1}{\sqrt{k}} > \sum_{k=1}^{n} 2\,(\sqrt{k+1}-\sqrt{k}) = 2\sum_{k=1}^{n}\sqrt{k+1}-\sqrt{k}$$
$$= 2\,(\underbrace{\sqrt{2}-\sqrt{1}}_{k=1}+\underbrace{\sqrt{3}-\sqrt{2}}_{k=2}+\underbrace{\sqrt{4}-\sqrt{3}}_{k=3}+\ldots+\underbrace{\sqrt{n+1}-\sqrt{n}}_{k=n})$$
$$= 2\,(-\sqrt{1}+\sqrt{n+1}) = 2\,(\sqrt{n+1}-1)$$

3) Puisque la suite des sommes partielles est non bornée, elle diverge.